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THE
ELEMENTS
OF
Linear Perspective
DEMONSTRATED BY
GEOMETRICAL PRINCIPLES,
And applied to the most general and concise
MODES OF PRACTICE.
WITH AN
INTRODUCTION,
CONTAINING SO MUCH OF THE
Elements of Geometry
AS WILL RENDER THE WHOLE
RATIONALE of PERSPECTIVE INTELLIGIBLE,
WITHOUT ANY OTHER PREVIOUS
MATHEMATICAL KNOWLEDGE.

By EDWARD NOBLE.

L O N D O N.

Printed for T. DAVIES, in Russel-Street, Covent-Garden,
Bookseller to the Royal Academy.

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V

THE
SIR JOSHUA REYNOLDS

P R E S E N T

LIBRARY OF THE

R/O. N. A. S. A. G. A. B. M. Y.

GEOMETRICAL PRINCIPLES

PAINTING

MODERN

FELLOW OF THE SOCIETY

IN THE



SIR

If the following Work should be a Means of rendering a Scientific Knowledge of Perspective more universal among the younger Practitioners of the Arts, and if the Foundation should contain the good Ought you are pleased to receive the Plan of it, it will more than recompense the Time and Gains the Society / Ambition of

THE

Your most obedient

and most humble

and most humble

and most humble

TO
SIR JOSHUA REYNOLDS, Knt.

P R E S I D E N T
OF THE
R O Y A L A C A D E M Y
O F
PAINTING, SCULPTURE and ARCHITECTURE,
AND
FELLOW OF THE ROYAL SOCIETY.

S I R,

IF the following Work should be a Means of rendering a Scientifical Knowledge of Perspective more universal among the younger Practitioners of the Arts; and if the Execution should confirm the good Opinion you was pleased to conceive from the Plan of it; it will more than recompence my Pains, and gratify the highest Ambition of,

S I R,

Your most obedient

and most humble Servant,

EDWARD NOBLE.

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lar page numbered on each, taking care to place
the upper edges of those which read long-ways,
outwards: but the best way will be to sew them
up by themselves, in which case they will be much
easier referred to.*

ERRATA in the Introduction.

Page xv. line 7. put the colon after it.

Page lxxv. line 1. read, and since B is the same multiple, part of

Page lxxvii. line 11. for all aliquot, read aliquot.

PREFACE.

P R E F A C E.

THOSE who are content with learning by rote the mechanical rules which compose the practice of an art, without any solicitation concerning the steps which reason pursued in their investigation, are more fitted for works of labour than of genius: however their cotemporaries may admire their industry, posterity will never hear of their improvements and inventions. In a word, they are totally void of that spirit of enquiry, and liberality of sentiment which, by exciting the mind to trace effects up to their causes, becomes the parent of discovery. It is this disposition to which we owe the perfection of the arts, and the extension of science; and which, with pleasure, is observed to characterize the most eminent of those whose pursuits lead them to stand in need of the art of perspective.

It is not wonderful, then, that those books which treat this art in a mechanical manner, should disgust the generality of their readers, or that the most encouragement and attention has been paid to those who have accompanied their precepts

precepts with demonstrations of their evidence: But as it must be allowed that the practical perspective is much more generally understood than the reasons on which it is founded, it follows, that either these demonstrations have been unintelligible for want of some *præcognita* with which the student is generally unacquainted, or else their evidence has been too feeble to engage attention, or establish remembrance: that each of these may have been the case, will appear by dividing the theoretical writers on perspective into two classes.

In the first class may be ranked those writers who have treated this subject in a liberal and masterly manner, by investigating its rules with that geometrical rigour, and applying them to practice with that universality, which distinguishes the works of mathematicians; but as they require some previous geometrical knowledge, with which most of those to whom perspective is more immediately necessary are unacquainted, they are unintelligible where they are most wanted, and consequently useless where they would be most prized.

In the second class I comprize those who endeavour to demonstrate the principles of perspective without having recourse to mathematical speculations. These gentlemen may be compared to an architect who would support a building whilst the foundation is removed: His skill and contrivance in a curious disposition and arrangement of props, might be worthy of commendation; but his absurdity would be equally ridiculous, was he to as-
sure

sure us his fabric was more durable, beautiful and convenient, because it stood upon crutches. I mean not, therefore, to depreciate the ingenuity of this class of writers, but cannot subscribe to their utility. Their mode of demonstration, by an infinity of pasteboard and string machinery, * bears the same relation to geometrical reasoning, which gestures and dumb shew do to verbal expression; and even this comparison is more honorary than just; for gesticulation is of real advantage where the organs of hearing or of speech are wanting; but if the understanding is deficient this pasteboard apparatus is useless, as more acuteness is required to comprehend its construction and application, than will be wanted in a train of geometrical reasoning. He who generously wishes not only to learn, but also to improve any mathematical art, should make himself master of those mathematical principles on which it is founded; otherwise he is as absurd as one who, being to settle in a foreign country, spends more time in inventing signs and gestures to make himself understood, than would render him master of the language of its inhabitants. Nor is a partial and confined idea of the art the only inconvenience that is obtained from studying this class of writers; for some of them, being deficient in mathematical

* This is not meant as a reflection on such ingenious apparatus as have been exhibited in lectures on this subject, (particularly by Mr. Thomas Malton) which are useful to give a general introductory notion of this art to beginners; and are highly entertaining to proficients.

knowledge, have fallen into errors which they have charged upon the art itself, and have accused perspective (whose utility they extol, and with whose charms they pretend to be enamoured) of imperfections which exist no where but in their own conceptions. Some of these will be taken notice of in the course of this work, not with any intention to gratify spleen, or to depreciate persons who, in other instances, must be allowed to have merit, but purely to vindicate an art founded on the immutable evidence of reason and geometry, from false imputations.

Dr. Brook Taylor is the most excellent, as well as the most concise writer on this subject. In the year 1715 he new modelled the art of perspective, and comprized it in a few pages elegantly deduced from geometrical principles: but his treatise being too speculative for the people for whose use he intended it, and the schemes being entirely void of ornament, which he afterwards thought made it appear still more dry and obscure; in 1719 he re-published it in a form somewhat different and enlarged; the plates also are more ornamental than those of the first edition. Yet his fate has been to be more admired and celebrated than understood. His work has been generally thought only fit for mathematicians. He sometimes quotes Euclid in his demonstrations, and thereby alarms those who know not that Euclid's Elements are built upon a few principles of common sense, without which the most domestic and simple negotiations of life cannot be transacted;
and

and that what they shun as subjects too sublime and intricate for their comprehension, are only the most familiar truths made artificial by regularity, and disguised by a technical language.

Perspective is one of the many arts which owe their being to geometry, and whose principles cannot be explained with so much ease, elegance and evidence by any arguments which are not derived from this divine science. But such a profound knowledge of geometry as cannot be obtained without intense application, and a very extensive capacity, is by no means necessary to understand the principles of perspective: on the contrary, they are easily deduced from a few of the most simple and useful propositions in Euclid's Elements, which will not require much exertion of the understanding to conceive or harrass the memory to retain them; and are as applicable to many other arts as this for which I have selected them in the introduction to this work.

To expatiate on the usefulness of geometrical knowledge, would require greater powers than I can exert, and is more unnecessary, as that task has been superseded by many elegant writers of every age and country. The mind, like the body, acquires strength from exercise. Prejudice and credulity are the great sources of error; but those who are accustomed to that exactness and impartiality which is exerted in geometrical enquiries, will not often condemn without examination, or adapt without proof. For these reasons some
know-

knowledge of the elements of geometry has ever been considered as a necessary appendage to a liberal education; and the little that composes the introduction to this work, is but what the young artist should be acquainted with, even if it had no relation to his business. Surely then, when at the same time it is the foundation of that art which shapes his outline, and makes his piece appear what he designs it should represent, he will not grudge the trouble which this little geometrical knowledge will cost him to attain.

I am not ignorant that the warmth of imagination and luxuriance of fancy, which impels the mind to the cultivation of the fine arts, is little captivated with abstracted and difficult speculations. It is essentially different from that calm and pen- sive turn of mind and strength of reason that, with silent enjoyment and inflexible industry, will trace an intricate truth through all its mazes and elusions, and exult in the discovery with a zest unknown to all but mathematicians: these are formed to explore the meanders of the vein, and are happy to dig the ore of science, whilst the others apply it to the convenience and ornament of life, by branching it into those arts, whose utility adds to our ease and safety; and whose polish enlivens our fancy, and engages our admiration.

To make, therefore, a book of science agree- able to an artist, the elementary part should not be all crowded together, but the irksomeness of abstruse investigation should be constantly re- lieved by pleasant and useful applications to prac- tice

P R E F A C E.

11

rice, that so without lessening the force of demonstration, it's rigour may be less observed by appearing in piecemeal.

For this purpose in the disposition of the following work, I have been careful to blend the Theory with the Practice. The Geometrical Introduction, extracted from Euclid, I have endeavoured to render more agreeable by concise accounts of the Use of each Proposition; and have pointed out the places wherein they are referred to in the demonstrations of the several Operations of Perspective, in order to convince the Tyro of their utility. The work then begins with an account of the Nature and Design of Perspective; this is followed by a little Theory from which a rude and simple method of Practice is deduced; and this, by a gradual enlargement of Theory, is at length improved to the most concise and general Modes of Practice hitherto discovered. The utility and defects of each method are also pointed out, and the latter are supplied, by considerations drawn from the Geometrical Introduction, in a more ample manner than is to be found in more Voluminous Works.

In order to render my reader perfectly master of the subject, I have, by way of Digressions, explained so much of the nature of Catoptrics, and the construction of the Eye, as will make him understand the manner in which Vision is performed; and will enable him to judge of several controverted points better than some writers seem to have done.

b

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P R E F A C E

In the choice of Examples I have studied use and simplicity more than intricacy and ostentation; and have laboured more to assist the skill of the learner than to display my own. Prisms on multiform Bases, and in every variety of situation, with their sides diversified with the likeness of doors and windows, and their tops shaped similar to the roofs of houses, are esteemed more proper to illustrate this subject than the Platonic Bodies, which are so much paraded with in most books on this subject, but which are as void of utility to the Artist, as their greek names are void of Harmony to the ear: nor is the double-cross so largely expatiated on in this, as in many works of this kind: but to make amends for this neglect, I have enlarged on points whose importance and difficulty with, I hope, apologize for prolixity.

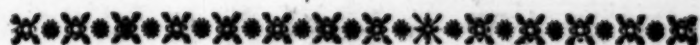
Ornament does not facilitate perspicuity; fine prints are valuable as pieces of art, but they are useless in books of science: If obscurity is avoided, beauty may be dispensed with, provided there is an adequate abatement in the price of the book: it is hoped this will suffice for want of elegance in the Plates, which were not intended to please the eye, but to assist the understanding.

A treatise on a subject which has been handled by a succession of men eminent for genius and invention, cannot abound with new Discoveries: claims to novelty must arise more from manner than matter: at the same time it is hoped nothing of Importance is omitted that can be found

in other works of this nature; and, that some things of consequence are explained which have hitherto been very little touched upon, if not wholly neglected.

It would be equally foolish and vain to suppose I have been so happy in my designs and execution as to preclude all just reprehension: or that there is no want of regularity in the arrangement of the work, no inelegance in the demonstrations, or no incorrectness of style, to entitle me to censure; perhaps in some places I am tediously prolix, and in others obscure through brevity: but as care has been taken to guard against these faults, it is to be hoped they do not so much abound as to deserve the condemnation of the candid and judicious, who know how much easier it is to discover faults than to avoid them.

P R E F A C E .



INTRODUCTION.

CONTAINING SUCH

Geometrical Propositions,

(Extracted from Euclid's Elements)

As are necessary to demonstrate

T H E

PRINCIPLES *of* PERSPECTIVE;



INTRODUCTION

It may be proper, before we set out, to apprise the learner, that the improvement he will derive from reading this book of Science, will be nearly proportionable to the Care and Attention with which he peruses it: that the following Rules will be read to most advantage in the order in which it is printed; and that if he draws the several Lessons on a larger scale than the book exhibits, the advantages will be more than compensated for the trouble.

Explanation of Characters.

This mark is called plus, and signifies that the two quantities are to be added together. Ex. $A + B$ means that the quantity denoted by A is to be added to that denoted by B.

This mark is called minus, and signifies that the quantity denoted by B is to be subtracted from the quantity denoted by A. Ex. $A - B$ means that the quantity denoted by B is to be subtracted from the quantity denoted by A.

INTRODUCTION.

IT may be proper, before we set out, to apprise the Learner, that the improvement he will derive from reading any Book of Science, will be nearly proportionable to the Care and Attention with which he peruses: it that the following Work will be read to most advantage in the order in which it is printed: and that if he draws the several schemes on a larger scale than are here exhibited the advantages he will derive therefrom will more than compensate for the trouble.

Explanation of Characters.

+ This mark is called *plus*, and when it stands between two quantities, it shews they are to be added together. Ex. $A+B$ is read *A plus B*, and means that the quantity represented by *A* is to be added to that represented by *B*.

—, This mark is called *minus*: and when placed between two quantities, it means that the quantity which follows it must be subtracted from that which precedes it. Ex. $A-B$ is read, *A minus B*; and means that the quantity represented by *B*, must be subtracted from that represented by *A*.

×, Is

$\times$, Is the mark for Multiplication, and when placed between two quantities it shews they are to be multiplied together. Ex. $A \times B$, shews the quantities represented by A and B are to be multiplied together.

When one quantity is to be divided by another quantity, we draw a line below the former, and write the latter underneath: as for example $\frac{A}{B}$ shews that the quantity represented by A, is to be divided by that represented by B.

This mark $=$ must be read "*equal to*" and signifies that the quantity, or quantities, which precede it are equal to those which follow it. Thus $A=B$ signifies that the quantity denoted by A is equal to that denoted by B.

To exemplify all this, suppose A signifies 12, and B signifies 3, that is $A=12$ and $B=3$. Then is

$$A+B=15$$

$$A-B=9$$

$$A \times B=36$$

$$\frac{A}{B}=4$$

Is the mark for Multiplication, and when placed between two quantities it shows they are to be multiplied together. Ex. $A \times B$, shows the quantities represented by A and B are to be multiplied together.

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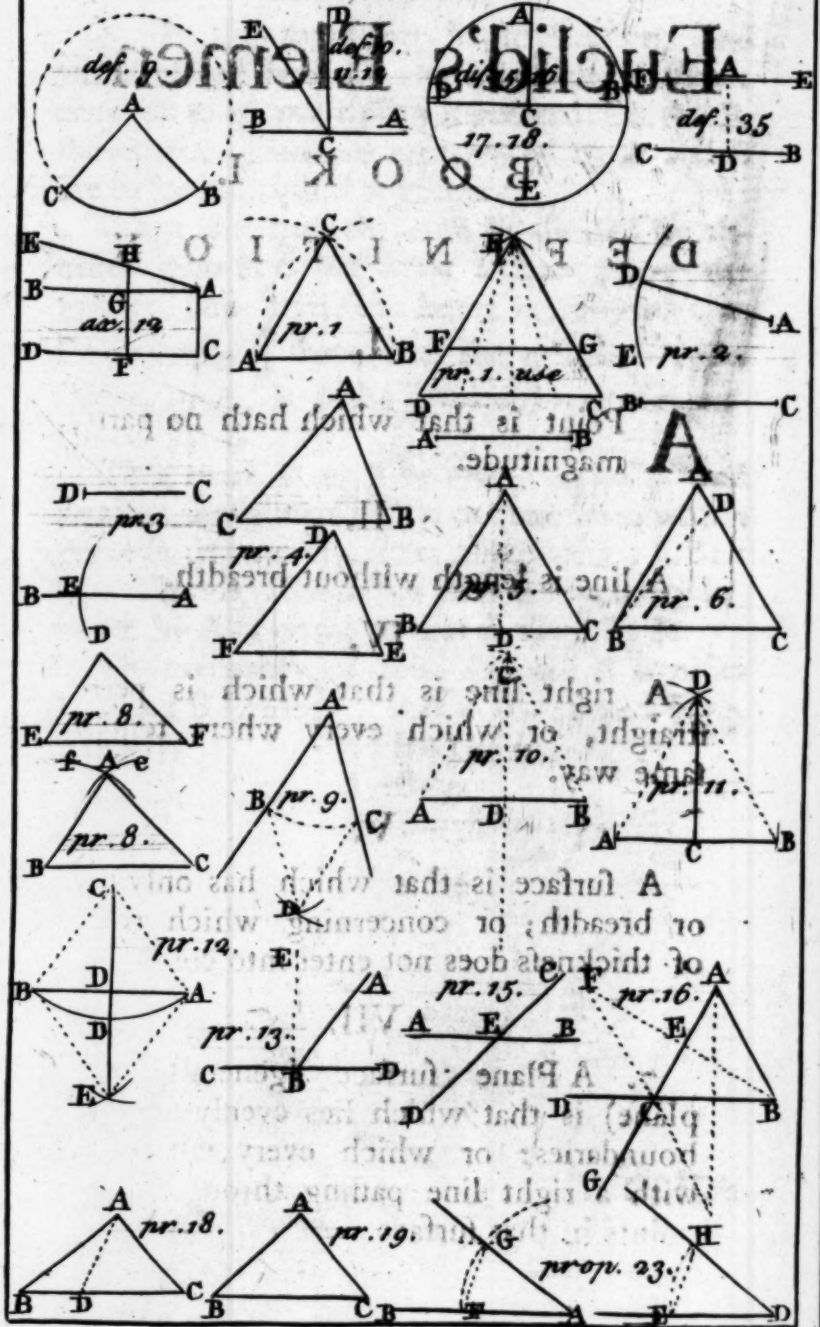
$$A + B = 15$$

$$A - B = 9$$

$$A \times B = 36$$

$$\frac{A}{B} = 4$$

Euclid Book 1.



Euclid's Elements.

BOOK I.

DEFINITIONS.

I.

A Point is that which hath no parts, or magnitude.

II.

A line is length without breadth.

IV.

A right line is that which is perfectly straight, or which every where tends the same way.

V.

A surface is that which has only length or breadth; or concerning which the idea of thickness does not enter into consideration.

VII.

7. A Plane surface (generally called a plane) is that which lies evenly between its boundaries: or which every where agrees with a right line passing through any two points in that surface.

VIII.

IX.

A right lined-angle (simply and generally called an angle) is the inclination, which two right lines meeting in a point have to each other.

An angle then is said to be greater or less according as the lines which form it are less or more inclined to each other: and this inclination is measured by describing a circle (BCD) about the angular point (A) as a center; and the arch (B C) intercepted between the lines forming the angle is its measure.

In order to ascertain the magnitude of the arch (B C) which measures an angle (B A C) or the proportion which it bears to the whole circumference of the circle (B C D). The said circumference is supposed to be divided into 360 equal parts called degrees: each of these is again divided into 60 equal parts called minutes: and each of these (when very great accuracy is required) is subdivided into 60 equal parts, called seconds. And the angle is said to be of so many degrees minutes, and seconds, as are contained in the arch intercepted between the lines which form it.

X.

If a right line (D C) standing upon another right line (A B) make the angles on each side of it (as A C D and B C D*) equal to

b 4

each

* When several angles are formed at the same point they are distinguished by three letters, the middle one, of which expresses the angular point, and the other two, the extremities of the lines forming the angle.

each other, each of those angles is called a *right angle*; and those lines (A B) and (C D) are said to be *perpendicular to each other*.

XI.

An *Obtuse Angle*, is that which is greater than a right angle.

Thus the angle A C E is obtuse because it is greater than the right angle A C D.

XII.

An *Acute Angle*, is that which is less than a right angle.

Thus the angle E C B is acute because it is less than the right angle D C B.

XIV.

A *Figure*, is any bounded space; and if it is formed on a plane surface, it is called a *plane figure*.

XV.

A *Circle* is a plane figure bounded by an uniformly curved line, called its *Circumference*, to which all lines drawn from a certain point within the figure are equal: which equal lines are called *Radii*.

XVI.

And that point is called the *center* of the circle.

XVII

XVII.

The diameter of a circle is a right line drawn through the center, and terminated at both ends by the circumference.

XVIII.

A semicircle is a plane figure bounded by a diameter, and that part of the circumference of a circle cut off by that diameter.

A B D E is the circumference; C its center; B C, A C and C D &c. are Radii: B D is a diameter: and the figure B E D is a semicircle.

XXI.

A plane figure bounded by three right lines, is called a triangle.

XXII.

Any plane figure bounded by four right lines is called a quadrangle.

XXIII.

Plane figures bounded by more than four right lines are called Polygons. Whereof those having five sides are called Pentagons, those having six sides are Hexagons, those having seven sides are Heptagons, those having eight sides are Octagons, &c.

If the sides and angles of a polygon are all equal, it is called a Regular Polygon; or a Regular

Regular Pentagon, Hexagon, Heptagon, &c.
according as it consists of five, six, or seven
sides, &c.

XXIV.

An Equilateral Triangle is that whose sides
are all equal.

XXV.

An Isosceles Triangle is that which hath two
equal sides.

XXVII.

A right angled Triangle is that which hath
one right Angle.

XXX.

A Square is a quadrangle, whose sides are
equal, and whose angles are all right angles.

XXXI.

A Rectangle is a quadrangle, whose angles
are all right angles, but whose sides are not
all equal.

XXXII.

A Rhombus is a quadrangle which has all its
sides equal, but its angles are not right angles.

XXXV.

Parallel Lines are such, which being situ-
ated in the same plane are every where equally
distant;

distant; or if infinitely produced both ways will never meet.

The former of these definitions of parallel lines implies the latter, but it lays us under the necessity of defining what is meant by the distance of a point from a line.

The distance of a point (A) from a line (B C) is a right line (A D) drawn from the given point (A) perpendicular to, and terminated by that line.

XXXVI.

A parallelogram is a quadrangle, whose opposite sides are parallel.

I have been obliged to follow the order of Euclid in his definitions, because of the quotations in the ensuing work, but it would have been better to have placed the 35th definition immediately after the 10th, and the 36th (which is omitted in Euclid) should have preceded the 30th; for a rectangle is only that species of a parallelogram, whose angles are all right: and a square is that particular kind of rectangle, whose sides are all equal. I shall conclude with what Euclid makes his 34th definition.

Every quadrangle which is not a parallelogram, is called a trapezium.

POSTULATES

POSTULATES.

1. That a right line may be drawn through any two points.
2. That a given right line may be continued or lengthened at pleasure from each end.
3. That about any point as a center, and with a radius equal to any given line, a circle may be described.

A X I O M S.

1. Things equal to one and the same thing are equal to each other.
2. If equal things be added to equal things, the sums will be equal.
3. If equal things be taken from equal things, the remainders will be equal.
6. Things that are double of the same thing, are equal to each other.
7. Things that are halves of the same thing, are equal to each other.
8. Such figures as being laid upon each other, every where agree, are equal: that is, the sides and angles of the one are respectively equal to the sides and angles of the other.

9. The

* A *postulate* is a petition for the learner's assent to the possibility of performing some very easy operation; the method of doing which is obvious to every one, or which every one immediately knows he can do without any instructions.

|| An *Axiom* is a truth so very apparent as to stand in no need of demonstration: but extorts an assent as soon as proposed by being self-evident. No truth, however certain, can be admitted as an *Axiom*, unless its evidence can be immediately understood without any argument to enforce it.

9. The whole is equal to all its parts taken together, and is greater than any part alone.

10. Two lines cannot include or bound a space.

12. If two right lines (A E and A B) meet in a point (A) they cannot be both parallel to another line C D. *This, tho' more plain than Euclid's 12th Axiom, is hardly self evident: wherefore let A C and F G be both perpendicular to C D. Then if A B is parallel to C D, we have F G = A C (by Def. 35.) also if A E is parallel to C D then is F H = A C. Wherefore (by Axiom 1.) F G = F H, a part to the whole, which is impossible; consequently if A B is parallel to C D, A E is not so.*

PROPOSITION † I.

Problem.

Upon a given right line (A B) to describe an equilateral triangle (A B C.)

1st. About A, as a center, with the distance A B, describe the arch of a circle B C (by post. 3.)

2d. About

† A *Proposition* is either an assertion of some property, whose truth is to be investigated; in which case it is called a *Theorem*. Or it is some practical operation proposed for performance; and then the *Proposition* is called a *Problem*, as in the case before us.

2d. About B, as a center, with the same distance A B, describe the arch A C cutting the former arch B C, in the point C.

3d. Draw A C and B C (*by post. 1.*) then is A B C, an equilateral triangle, whose sides are all equal to A B. Q. E. F.

DEMONSTRATION.

For because A is the center of the arch B C, A C is equal to A B (*by Def. 15.*) And because B is the center of the arch A C, B C is also equal to A B (*by Def. 15.*) wherefore A C and B C being both equal to A B, they are equal to each other (*by Ax. 1.*) consequently the sides A C, B C and A B being equal, the triangle A B C is equilateral (*by Def. 24.*)

This problem is used to demonstrate several other propositions, It also furnishes us with a method of dividing a given line into any number of equal parts.

Thus to divide the line A B into 4 equal parts: set off on any other line C D 4 equal parts from C to D; and on C D make an equilateral triangle, C E D, and from the vertical angle C E D draw lines to the divisions on C D; then will this triangle, C E D, serve us to divide any line, that is less than C D, into 4 equal parts: thus take the distance A B in the compasses and lay it from E on both sides E C and E D to F and G, then draw F G, which will

be equal to AB and will be divided into 4 equal parts by the lines drawn from the vertical angle E to the divisions on CD , as the figure sheweth.

The reason of all this will appear when the reader understands the propositions we have extracted from the 6th book.

PROPOSITION II.

Problem.

At a given point (A) to put a right line equal to a given right line (BC .)

1. At the given point A as a center, with a radius equal to the given line BC describe the arch of a circle DE (by post. 3.)

2d. Draw a line (AD) from the center A to the circumference DE of this arch, then shall this line AD be equal to the given line BC .

DEMONSTRATION.

It is evident from this construction and def. 15. that all lines drawn from the center A to the circumference DE , must be equal to the given line BC , consequently AD must be equal to BC . Q. E. D.

This problem is as necessary in almost every practical operation as its solution is easy.

PROPOSITION III.

Problem.

Two unequal lines (AB and CD) being given; to cut off from the greater (AB) a part (AE) equal to the less (CD .)

At

At the extremity (A) of the greater line (A B) as a center, with a radius equal to the less, (C D) describe the arch of a circle (by post. 3.) cutting A B in E; then is A E a part of A B, equal to C D. Q. E. F.

This is too evident to need demonstration. This problem is as useful and extensive in application as the preceding.

PROPOSITION IV.

Theorem.

If there are two triangles (A B C) and (D E F) having two sides (A B) and (A C) and their contained angle (A) in the one, respectively equal to two sides (D E) and (D F) and their contained angle (D) in the other; then will those triangles be equal in all respects: that is, the remaining side (B C) in the one, will be equal to the remaining side (E F) in the other, and the remaining angles (B and C) in the one triangle, will be equal to the remaining angles (E and F) in the other; each to each, which are opposite to equal sides.

DEMONSTRATION.

Conceive the point D, to be laid on the point A, and the line D E, upon the line A B; then, because these lines are equal, the point E will fall upon B; also because the angle D, is equal to the angle A, the line

D F

D F will coincide with A C; and since these lines are equal (*by supposition*) the point F will fall on C, wherefore the line E F coincides with the side B C, and the two triangles (A B C and D E F) agree in all respects (*Vide Ax. 8.*) Q. E. D.

This proposition is of use to demonstrate the next, as well as many others, and will be wanted in page 152 of the ensuing work.

PROPOSITION V.

Theorem.

*The angles (B and C) at the base * (B C) of an isosceles triangle (A B C) are equal.*

DEMONSTRATION.

Conceive the line A D, to be so drawn as to divide the angle at A into the two equal parts B A D and D A C, then because B A is equal to A C (*by supposition*) and the side A D is common, the two triangles B A D and D A C will have two sides B A and A D, and the included angle B A D, in the one triangle, equal to the two sides A C and A D, and the included angle D A C, in the other triangle: wherefore the corresponding angles C and B (which are opposite to the equal sides A B and A C) are equal (*by prop. 4.*)

PROPOSITION

* The base of any figure is that side on which it is supposed to stand.

PROPOSITION VI.

Theorem.

If two angles ($A B C$ and $A C B$) of a triangle ($A B C$) are equal, the sides ($A B$ and $A C$) opposite to them will be equal.

DEMONSTRATION.

For if the sides $A B$ and $A C$ are not equal let one of them, as $A C$, be the biggest, from which cut off a part, $C D$ equal to $A B$.
(by prop. 3.)

Then in the two triangles, $A B C$ and $D B C$, the two sides $A B$ and $B C$, with the included angle $A B C$, in the one triangle ($A B C$) is equal to the two sides $D C$ and $B C$, and the included angle $D C B$ in the other triangle ($D B C$) wherefore (by prop. 4.) those two triangles ($A B C$ and $D B C$) are equal in all respects, which is impossible, therefore the sides $A B$ and $A C$ cannot be unequal. Q.E.D.

Corol.

Hence if all the angles of a triangle are equal, all the sides must be equal, and the triangle is equilateral (Def. 24.)

This proposition will be wanted to demonstrate the 19th. Also by it, and the 32d we know that if one angle of a right angled triangle is equal to 45° . the other is also equal to 45° . and the

two sides are equal, and thence we have an easy way to measure the height of a building AB , for if we take a quadrant and go from the building till the plumb-line cuts 45° as in the point C , the eye is then at the same horizontal distance BC as is equal to the height of the building AB above the eye.

PROPOSITION VIII.

Theorem.

If the three sides of one triangle (ABC) are respectively equal to the three sides of another triangle (DEF) the corresponding angles (A and D , B and E , C and F) which are opposite to the equal sides (AB and DE , BC and EF , AC and DF) will be equal, and the triangles will be equal in all respects.

DEMONSTRATION.

On the point B , as a center, with the radius equal to BA , describe an arch $* A$ through the point A (by post 3.) also on C as a center with the radius CA , describe another arch (Ae) passing through the same point A . Suppose now the line EF to be laid on the line BC they will agree, because they are supposed equal; also because DE is equal to AB , the point D must fall somewhere in the arch

* An Arch, or an Arc is any part of a circumference of a circle.

arch Af , (by Def. 15.) and because DF is equal to AC , the said point D must fall somewhere in the arch Ae ; seeing then the point D falls in both the arches Ae and Af , it must fall on the point A , where those arcs intersect each other, and the triangles must agree in all respects (by Ax. 8.) Q. E. D.

This proposition will enable us to perform the 2^d. Also, when we are unprovided with any instrument to measure an angle we can, from this consideration, take the plot of a field, &c. by measuring its sides and diagonal.

PROPOSITION IX.

Problem.

To bisect, or divide a given angle (BAC) into two equal parts (BAD and DAC .)

At the angular point A , as a center, with any radius at pleasure describe an arch (BC) cutting the sides of the given angle in the points B and C . (by post. 3.)

At each of these points, as centers, with the same radius, describe arcs, cutting each other in D .

Lastly, draw AD , which will bisect the given angle BAC , as required.

DEMONSTRATION.

In the two triangles BAD and DAC , we have $AB=AC$; $BD=DC$ and the side AD common to both, wherefore seeing the three

sides of one triangle, BAD , are respectively equal to the three sides of the other triangle DAC , the corresponding angles BAD and DAC (which are opposite to the equal sides BD and DC) are equal, (*by prop. 8.*) and because these equal angles together, compose the given angle BAC , that angle is bisected. *Q. E. F.*

This is a most useful problem in practical geometry, as it helps us to perform many others: also since by this method of bisecting an angle, we also bisect the arch, which measures that angle, we can, by continual bisections, divide a quadrant into eight equal parts, whereby the whole circumference of the circle will be divided into 32 equal parts as in the nautical card, or compass. We shall also refer to it in pages 207 and 208.

PROPOSITION X.

Problem.

To cut a given right line (AB) into two equal parts.

SOLUTION.

1st. On the AB make an equilateral triangle ABC . (*by prop. 1.*)

2d. Draw CD bisecting the angle ACB (*by prop. 9.*) which will cut AB in the two equal parts AD and DB . *Q. E. F.*

DEMON-

DEMONSTRATION.

For because AC is equal CB , and the angle ACD = the angle DCB . In the two triangles ACD and DCB , we have two sides (AC and CD) and the included angle (ACD) of one triangle, equal to the two sides (BC and CD) and the included angle (DCB) of the other triangle, wherefore the corresponding sides AD and DB are equal. (by prop 4.)

This problem is of great use in the solution of many others, and the preparation of many theorems for demonstration. It is also quoted in page 223.

PROPOSITION XI.

Problem.

To draw a right line (CD) perpendicular to a given indefinite right line (AB) from any point (C) given in the same.

SOLUTION.

1st. At the given point C , with any radius, describe arcs, cutting the given line in the points A and B .

2d. At these points (as centers) with any radius larger than AC or BC , describe arcs cutting each other in D .

3d. Draw DC , which is the perpendicular sought.

DEMON-

DEMONSTRATION.

For (by construction) $AC = CB$, and $AD = DB$; wherefore in the two contiguous triangles ACD and DCB , the three sides AD , AC and DC of the one; are respectively equal to the three sides DB , BC and DC of the other: consequently the contiguous and corresponding angles ACD and DCB are equal (by prop. 8.) and CD is perpendicular to AB (by def. 10.)

The utility of this problem extends to the most domestic purposes of life. The references to it in the ensuing treatise are too many to be recited here: the solution here given is chiefly useful when the given point is not very near the end of the given line, for when it is so, and we have not room to continue it, recourse must be had to a method deduced from the 31st proposition of the 3d book.

PROPOSITION XII.

Problem.

To draw a line (CD) perpendicular to a given indefinite line (AB) from a given point (C) out of that line (AB .)

SOLUTION.

1st. At the given point C , as a center, with any radius (that will reach beyond the given line AB) describe an arch ADB , cutting the given line AB in the points A and B ,
(by

(by *post* 3.) and on these points as centers, with any radius, describe arcs cutting each other in E. $\angle C = \angle A$

2d. Draw a line through the points C and E (*post*. 1.) which will be the perpendicular required.

DEMONSTRATION.

It is evident from this construction, that the two contiguous triangles A C E and B C E have the three sides A C, A E and C E, of the one, respectively equal to the three sides B C, B E and C E of the other, wherefore the angle A C E = angle B C E (*by prop*. 8.)

Again the two contiguous triangles A C D and D C B, having two sides A C and C D with the included angle A C D, in the one triangle, equal to the two sides B C and C D, and the included angle B C D in the other; the adjacent and corresponding angles C D A and C D B are equal, and C D is perpendicular to A B (*by def*. 10.)

This problem will be very often wanted in demonstrating the principles of perspective: its uses are almost as manifold as the preceding: also the solution here given has the same restraint, and admits of the same remedy as the last, which will appear from prop. 31. book 3.

PROPOSITION XIII.

Theorem.

If one line (A B) stands upon another line (C D) they make two angles (A B C and A B D,) which are either two right angles, or taken together are equal to two right angles.

If A B is perpendicular to C D the angles A B C and A B D are right angles (by def. 10.)

If A B is not perpendicular to C D draw (by 1. 11.) B E perpendicular to B D, then it is evident the angle A B C is greater than a right angle (C B E) by the angle E B A; also the angle A B D is less than a right angle (E B D) by the same angle E B A, so that the excess of the one angle (A B C) exactly balances the deficiency of the other (A B D) and renders their sum equal to two right angles. Q. E. D.

This theorem will be wanted in demonstrating many others, also when we know either of the angles A B D or A B C, we can find the other by subtracting the given angle from 180° . Thus if $A B D = 45^\circ$ the angle A B C is $= 135^\circ$.

PROPOSITION XV.

Theorem.

If two lines (A B and C D) cut one another, the opposite angles (A E C and D E B) shall be equal.

For

For because CE stands upon AB , the angles $CEA + CEB =$ two right angles (by 1. 13.) also because BE stands upon CD the angles $CEB + BED$ are equal to two right angles; wherefore $AEC + CEB = CEB + BED$ (by ax. 1.) take away the common angle CEB and there remains $AEC = BED$ (by ax. 3.) $Q.E.D.$

COROLLARY I.

If two lines cut each other, the four angles which they make at the point where they cut are together equal to four right angles.

COROLLARY II.

All the angles made by any number of lines meeting in one point, are together equal to four right angles.

This theorem is of use to demonstrate the next, and a great many others, in all parts of the mathematics.

PROPOSITION XVI.

Theorem.

If one side (BC) of a triangle (ABC) is produced, the exterior angle, ACD , is greater than either of the interior opposite angles (B or A).

Bisect AC in E (by 1. 10.) draw BE and produce it to F , and make $EF = BE$, also join FC .

Because AE and BE are respectively equal to CE and EF , (by construct.) and the included

cluded angle $AEB =$ included angle CEF (by 1. 15.) therefore the angle ACF or $ECF =$ angle A (by 1. 4.) But ACD is greater than ACF , wherefore it is greater than the angle A .

In the same manner if BC is bisected, and AC is produced to G , it may be proved that the angle BCG or its equal ACD (vide 1. 15.) is greater than the angle B .
Q. E. D.

This proposition will be wanted to demonstrate the 18th and 27th: it is also referred to in page 152. It might have been made a corollary to the 32d, were it not that it is wanted in the 27th on which the demonstration of the 32d depends.

PROPOSITION XVIII.

Theorem.

In any triangle (ABC) the greater angle is opposite to the greater side; to wit, if the side BC is greater than AC the angle A is greater than the angle B .

DEMONSTRATION.

Since (by supposition) BC is greater than AC cut off (by *prop.* 3.) a part (CD) of $BC = AC$.

Then because $AC = CD$ the angle $DAC = ADC$ (by *prop.* 5.)

But

included angle AEF = included angle CEH (by 1. 12) therefore the angle ACH or ECF = angle A (by 1. 4). But ACD is greater than ACE , wherefore it is greater than the angle A .

In the same manner if BC is bisected, and AC is produced to G , it may be proved that the angle BGC or its equal ACD (vide 1. 12) is greater than the angle B .

Q. E. D.

This proposition will be required to demonstrate the 18th and 19th. It is also required in page 12. It might have been made a corollary to the 2d, were it not that it is required in the 18th on which the demonstration of the 2d depends.

PROPOSITION XVIII.

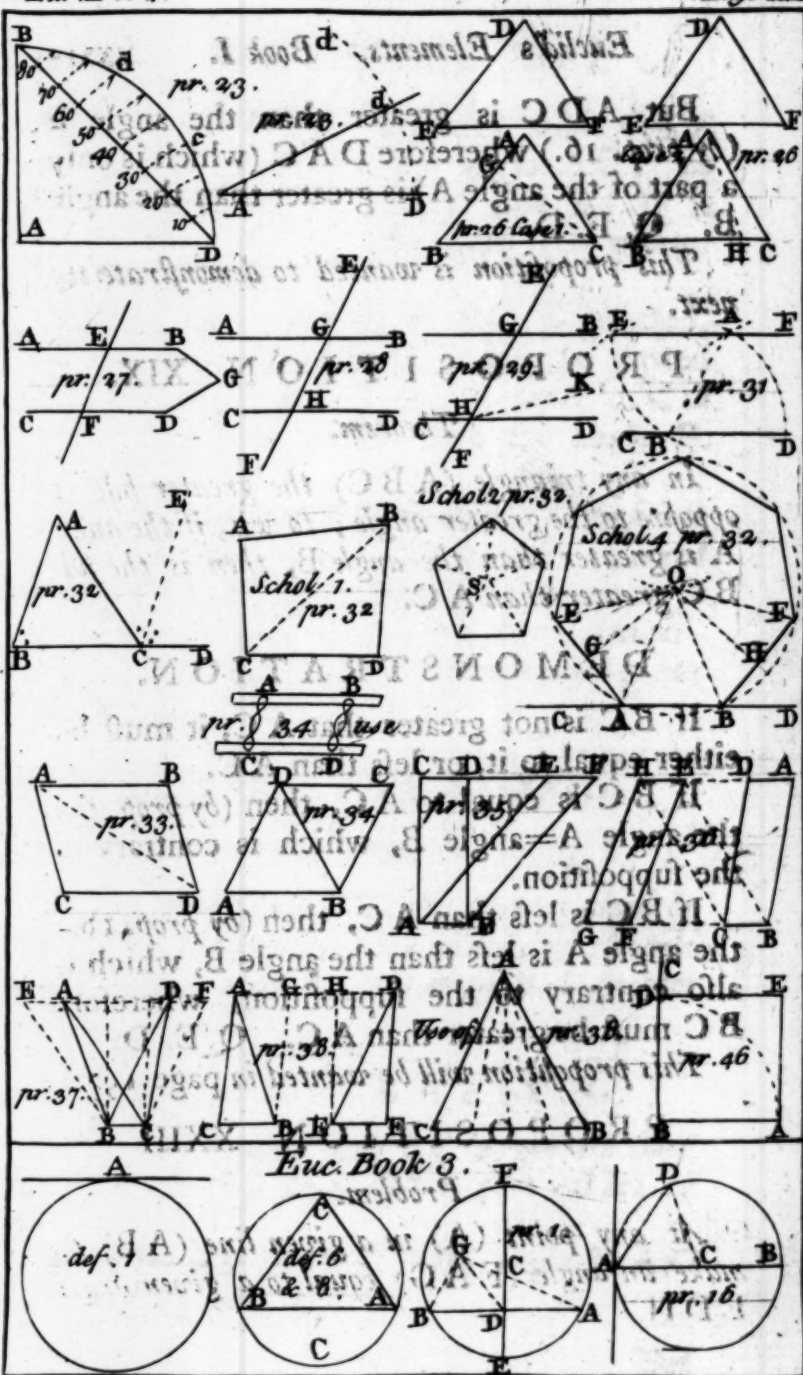
Theorem.

In any triangle (ABC) the greater angle is opposite to the greater side; to wit, if the side BC is greater than AC the angle A is greater than the angle B .

DEMONSTRATION.

Since (by supposition) BC is greater than AC cut off (by prop. 3) a part (CD) of BC = AC . Then because $AC = CD$ the angle BAC = angle BDC (by prop. 1).

||



But $\angle ADC$ is greater than the angle B (by prop. 16.) wherefore $\angle DAC$ (which is only a part of the angle A) is greater than the angle B . Q. E. D.

This proposition is wanted to demonstrate the next.

PROPOSITION XIX.

Theorem.

In any triangle (ABC) the greater side is opposite to the greater angle; to wit, if the angle A is greater than the angle B , then is the side BC greater than AC .

DEMONSTRATION.

If BC is not greater than AC , it must be either equal to it, or less than AC .

If BC is equal to AC , then (by prop. 6.) the angle $A =$ angle B , which is contrary to the supposition.

If BC is less than AC , then (by prop. 18.) the angle A is less than the angle B , which is also contrary to the supposition; wherefore BC must be greater than AC . Q. E. D.

This proposition will be wanted in page 152.

PROPOSITION XXIII.

Problem.

At any point (A) in a given line (AB) to make an angle (FAG) equal to a given angle EDH .

At

At the angular point (D) of the given angle (E D H) as a center with any radius (D E) describe an arch cutting its sides in the points E and H. Also with the same radius from A, as a center, describe the arch F G, cutting the given line (A B) in F. Lastly, open the compasses to the distance E H, and setting one foot in F with the other cut the arch F G in G, and draw A G, then shall the angle F A G = angle E D H.

DEMONSTRATION.

For if the lines E H and F G are drawn, it is evident, from the construction, that the three sides of one triangle (D E H) are respectively equal to the three sides of the other, A G F; and therefore $E D H = F A G$ (by 1. 8.)

This problem is of vast service in all parts of the mathematics; it will be wanted in the 31st of this book and in the 33^d of the 3^d book. In the practice of perspective we often want to make one angle equal to another. But in this solution we are supposed to have the given angle actually drawn out; whereas it frequently happens we have only given the number of degrees, &c. of which the angle to be drawn must consist; in this case we must use the line of chords, whose construction and use is as follows: Draw two lines A B and A D perpendicular to each other (by prop. 11.) at A, as a center with any radius describe the quadrant B D, and draw the line B D; then with the same opening set the compasses

compasses in B and D, and cut the arch of the quadrant in d and c, which by this means will be divided into three equal parts; divide (by trials) each of these parts into three equal parts, which will contain 10° , each of which being bisected (by prop. 9.) the arch B D will be divided into 18 equal parts of 5° each. Lastly, each of these divisions being again divided into 5 equal parts (by trials) the whole quadrant will be divided into 90 degrees.

Set one foot of the compasses in D and open the other to the several divisions on the arch B c D, and transfer them to the chord B D, which by this means will be divided into 90 unequal parts, and may be transferred to any straight line drawn on a ruler or scale.

If A D is a given line and we want to make an angle of 25° at the point A: take from the line of chords an opening of 60° , and set one foot in A, and with the other describe an arch (D d) then take 25° from the line of chords, and setting one foot in D with the other cut the arch D d in e. Lastly, draw A e, so shall D A e be an angle of 25° .

PROPOSITION XXVI.

Theorem.

If two triangles (A B C and D E F) have two angles (B and C) in one (A B C) equal to two angles E and F in the other triangle (D E F) and one side equal to one
d
side;

side; viz: either the sides (BC and EF) adjacent to the equal angles, or the sides (AB and DE) which are opposite to equal angles in each; then shall the other corresponding sides be equal, each to each, also the remaining angles (A and D) shall be equal.

C A S E I.

When those sides (BC and EF) are equal, which are adjacent to the angles which are equal.

1st, If AB is not equal to DE, one of them must be greater: then (by prop. 3.) cut off from the greater (as AB) a part (BG) equal to the less DE. Then because BG and BC are respectively equal to DE and EF, and the angle B = angle E (by supposit.) the angle BCG = angle F (by prop. 4.) But the angle F = BCA (by supposit.) wherefore the angle BCG = angle BCA (by ax. 1.) but this is impossible, because the less cannot be equal to the greater; wherefore AB cannot be unequal to DE: since then AB and BC are equal to DE and EF, and the included angle B is equal to the included angle E. The triangles ABC and DEF are equal in all respects (by prop. 4.)

C A S E II.

When the sides (AB and DE) which are opposite to equal angles C and F in each triangle, are equal.

First

First, I say $BC = EF$, for if it is not let BC be the greater, and (*by prop. 3.*) make $BH = EF$.

Then because AB and BH , with the included angle B , are respectively equal to DE and EF , with the including angle E (*by sup.*) the angle $AHB = \text{angle } F$ (*by prop. 4.*) but the angle $C = \text{angle } F$ (*by sup.*) wherefore the angle $AHB = \text{angle } C$ (*by ax. 1.*) that is the exterior angle (AHB) of a triangle (AHC) is equal to its interior and opposite angle (C) which is impossible (*by prop. 16.*) wherefore BC is not unequal to EF .

Since then AB and BC are respectively equal to DE and EF , and the included angle $B = \text{included angle } E$: the triangles ABC and DEF are equal in all respects (*by prop. 4.*)

We shall have occasion for this theorem in page 19. 21. 22. and 175.. It is also particularly useful in surveying.

PROPOSITION XXVII.

Theorem.

If a right line (EF) falling upon two other right lines (AB and CD) makes the alternate angles AEF and EFD equal to one another; these two lines (AB and CD) shall be parallel.

DEMONSTRATION.

For if AB and CD are not parallel they will, if produced, cut each other: suppose then they cut each other, when produced, in G . Then will $EF G$ be a triangle, whereof the outward angle $A E F$ is greater than the inward opposite angle $E F G$ (*by 1. 16.*) contrary to the supposition by which those angles are equal: consequently the lines AB and CD cannot meet when produced towards B and D ; and the same argument will prove they cannot meet if produced towards A and C ; seeing then they can never meet, they are parallel (*by def. 35.*) Q. E. D.

This theorem will be wanted to demonstrate the next and to perform prop. 31. and is referred to page 33.

PROPOSITION XXVIII.

Theorem.

If a straight line (EF) falling upon two straight lines (AB and CD) makes the exterior angle (EGB) equal to the interior and opposite angle ($EH D$) upon the same side of the line; or makes the sum of the two interior angles ($BGH + GHD$) upon the same side equal to two right angles; those two straight lines (AB and CD) shall be parallel to each other.

DEMON-

DEMONSTRATION.

Since we suppose $EGB = GHD$, and because (by 1. 15.) $EGB = AGH$, it follows (by ax. 1.) that $GHD = AGH$, but these are alternate angles, wherefore AB is parallel to CD (by 1. 27.) Again if $BGH + GHD = 2$ right angles, and because $BGH + AGH = 2$ right angles (by 1. 13.) It follows that $BGH + GHD = BGH + AGH$ (by ax. 1.) wherefore, taking away BGH from both sides, we have left (by ax. 3.) $GHD = AGH$. And these being alternate angles, it follows that AB and CD are parallel (by 1. 27.) Q. E. D

This proposition will be wanted in pages 23 and 163.

PROPOSITION XXIX.

Theorem.

If a right line (EF) falls upon two parallel lines (AB and CD) then,

1st, The alternate angles (AGH and GHD) are equal.

2d, The exterior angle (EGB) is equal to the interior opposite angle (GHD), and

3d, The sum of the two interior angles ($BGH + GHD$) is equal to two right angles.

DEMONSTRATION.

1st, If AGH is not equal to GHD , make (by 1. 23.) $GKH = AGH$, then because these

d 3

are

are alternate angles HK is parallel to AB (by 1. 27.) and so is CD (by supposit.) which is impossible (by ax. 12.) consequently AGH is not unequal to GHD .

2d, Since $AGH = GHD$, and since $AGH = EGB$ (by 1. 15.) it follows (by ax. 1.) that $GHD = EGB$.

3d, Since $GHD = EGB$, we have (by ax. 1.) $GHD + BGH = EGB + BGH$. But $EGB + BGH$ is equal to two right angles; wherefore $GHD + BGH$ is also equal to two right angles. Q. E. D.

From this theorem we demonstrate prop. 32. also it is referred to in pages 22. 31. and 43.

PROPOSITION XXXI.

Problem.

Through any given point (A) to draw a line (EF) parallel to a given line (CD).

From the given point (A) draw a line (AB) cutting (CD) in any point (B) and make (by 1. 23.) the angle $EAB = ABD$; then shall EF be parallel to CD , because the alternate angles are equal (by 1. 29.)

The use of parallel lines is as frequent as perpendiculars; particularly in perspective they are so very often wanted, that the square and parallel ruler are almost indispensable for persons who make perspective representations, as by them more than nine-tenths of the labour which would arise

arise by working according to this proposition, and the tenth, is saved.

PROPOSITION XXXII.

If any side (BC) of a triangle (ABC) is produced, the exterior angle (ACD) is equal to the sum (B+A) of the two interior and opposite angles: and the sum of the three angles of any triangle is equal to two right angles, or 180 degrees.

DEMONSTRATION.

Through C draw (by 1. 31.) CE parallel to AB; then (by 1. 29.) the alternate angles A and ACE are equal; also the angle ECD is to the inward opposite angle B: consequently (by ax. 2.) $A+B=ACE+ECD=ACD$ (by ax. 9.)

Again (by 1. 13.) $ACD+ACB$ is equal to two right angles, wherefore (by ax. 1.) $A+B+ACB$ is equal to two right angles, which last are the three angles of the triangle ABC.

COROLLARY I.

Hence any angle of an equilateral triangle is one-third of two right angles or 60°.

COROLLARY II.

If the sum of two of the angles of any triangle, is equal to the sum of two of the angles of any other triangle, the remaining angle in the one is equal to the remaining angle in the other.

COROLLARY III.

If one angle of a triangle is a right angle, the other two taken together is equal to a right angle; or the one is the complement* of the other.

COROLLARY IV.

If the two sides of a right angled triangle are equal, the oblique angles are each equal to half a right angle or 45° (vide prop. 5.)

SCHOLIUM I†.

The sum of the four inward angles of any quadrangle (ABCD) is equal to four right or 360° .

For if the diagonal CB is drawn, the quadrangle will be divided into two triangles (CAB and CDB) the sum of whose angles are equal to the sum of the angles of the quadrangle, and also to four right angles.

SCHO.

* The complement of an arch or an angle is what it wants of 90 degrees.

† A scholium is a remark on some preceding proposition.

SCHOLIUM II.

The sum of the interior angles of any right lined plane figure is equal to twice as many right angles abating four, as that figure hath sides.

For if from any point (S) within the figure, lines are drawn to its angular points; there will be as many triangles formed as the figure has sides. But the angles of these triangles (which are equal to twice as many right angles) exceed the angles of the figure, by those which are formed about the point S, which (by 1. 15. cor. 2.) are equal to four right angles; whence the proposition is manifest.

SCHOLIUM III.

If the sides of any right lined plane figure are produced, the sum of the exterior angles will be equal to four right angles or 360° .

It is evident that the exterior and interior angles taken together will make twice as many right angles as the figure has sides; but the interior angles alone make that number of right angles abating four, wherefore the exterior angles are equal to those four right angles.

SCHOLIUM IV.

Hence we have a method of constructing a regular polygon of any number of sides, each of which shall be equal to a given line.

For

For as the interior angles of a regular polygon are equal (*by def. 23.*) the exterior angles must be also equal among themselves; wherefore 360° divided by the number of sides of which the polygon is to consist, gives the degrees contained in its exterior angle; by this means I find,

In a regular pentagon }
the exterior ang. } $= \frac{360^\circ}{5} = 72^\circ$

In a regular Hexagon, $= \frac{360^\circ}{6} = 60^\circ$

Heptagon, $= \frac{360^\circ}{7} = 51^\circ 26'$ nearly.

Octagon, $= \frac{360^\circ}{8} = 45^\circ$

Decagon, $= \frac{360^\circ}{10} = 36^\circ$, &c.

Let A B be a line on which it is required to construct a regular heptagon. Produce the given line A B both ways to C and D, and draw E A and B F each $=$ A B, and making the angles F B D and E A C each $= 51^\circ \frac{1}{2}'$ (nearly) also draw (*by E 1. 10. and 11.*) G O and H O bisecting B F and E A at right angles, and cutting each other in O. Lastly, at O as a center with the radius E O, A O, B O, or F O, (which are all equal) describe a circle, in whose circumference the line A B may be repeated

repeated exactly seven times; by which means the angular points of the regular heptagon will be obtained.

Various other advantage are derived from this proposition, whose use runs through all the mathematical arts: in the ensuing treatise it will be quoted in pages 16, 83, 101 and 152.

PROPOSITION XXXIII.

Theorem.

Right lines (A C and B D) joining the extremities of two equal and parallel lines (A B and C D) towards the same parts, are also equal and parallel.

DEMONSTRATION.

Draw A D, then because A B is parallel to C D (*by supposition*) the angle D A B is equal to the alternate angle A D C (*by prop. 29.*) Wherefore in the two triangles B A D and A D C, we have two sides A B and A D, and the included angle D A B respectively equal to the two sides C D and A D and the included angle A D C; wherefore (*by prop. 4.*) B D = A C, and the angle C A D = A D B: but these are alternate angles formed by the line A D cutting the two lines B D and A C, which therefore (*by prop. 27.*) are parallel as well as equal. Q. E. D.

This proposition will be quoted page 261.

PROPO-

PROPOSITION XXXIV.

Theorem.

The opposite sides and angles of a parallelogram are equal and a diagonal line divides it into two equal parts.

For if $A B C D$ is a parallelogram, and a diagonal $D B$ is drawn (*vide def. 36.*) Then in the two contiguous triangles $A B D$ and $B D C$, we have the angle $A D B =$ angle $D B C$ (*by 1. 29.*) also angle $B D C =$ angle $A B D$ (*by 1. 29.*) and the side $D B$ between the equal angles is common. Wherefore (*by 1. 26.*) $A B = D C$ and $A D = B C$ and the angle $A =$ angle C , and the triangle $A B D =$ triangle $B D C$. Q. E. D.

It would be easy to demonstrate the converse of this proposition; viz. That if the opposite sides of a quadrangle are equal, they are also parallel, and from hence arises the construction of that very useful instrument called a parallel ruler. This is composed of two flat rulers ($A B$ and $C D$) connected together by two equal brass-bars ($A C$ and $B D$) moving round the centers A, C, B and D , which with the intercepted parts of the rulers in all positions inclose a quadrilateral space, whose opposite sides are equal, and consequently parallel. — A single view of this instrument will dictate its use and application. The practice of perspective would be intolerably tedious were we obliged to have recourse to prob. 31. every time we want to draw one line parallel to another.

PROPO-

PROPOSITION XXXV.

Theorem.

Parallelograms (ABCD and ABEF) standing on the same base (AB) and between the same parallels (AB and CF) are equal to each other.

DEMONSTRATION.

For (by 1. 34.) $CD = AB$, also $EF = AB$ whence (by ax. 1.) $CD = EF$; add DE to each, and we have $CE = DF$, also $AC = BD$ (by 1. 34.) and the angle $ACF = BDF$ (by 1. 29.) wherefore the triangle $ACE =$ triangle BDF , (by 1. 4.) take away the common triangle DGE and the trapezium $CAGD =$ trapezium $BFE G$ (by ax. 3.) add the common triangle AGB to each, and the parallelogram $ABCD =$ parallelogram $ABEF$. Q. E. D.

This proposition will be wanted to demonstrate the next. It is also particularly useful in mensuration of land or other superficies by teaching to measure a parallelogram, whose angles are oblique; for the parallelogram $ABEF$ is equal to the rectangle $ABCD$, whose content is found by multiplying its base AB by its altitude AC .

PROPOSITION XXXVI.

Theorem.

Parallelograms (ABCD and EFGH) standing on equal bases (BC and FG) and between the same parallels are equal to each other.

DEMON-

D E M O N S T R A T I O N.

Draw BE and CH . Then because $BC = FG$ (*by supposition*) and $FG = EH$ (*by 1. 34.*) $BC = EH$ (*ax. 1.*) also BC is parallel to EH (*by supposition*) whence (*by 1. 33.*) BE is parallel to CH , and $BCEH$ is a parallelogram equal both to $ABCD$ and also to $EF GH$ (*by 1. 35.*) wherefore $ABCD = EFGH$. Q. E. D.

This proposition will be wanted to demonstrate the 38th.

P R O P O S I T I O N XXXVII.

Theorem.

Triangles (ABC and DBC) standing on the same base (BC) and between the same parallels (AD and BC) are equal to each other.

D E M O N S T R A T I O N.

Through B , draw BE parallel to AC , and through C , draw CF parallel to BD .

Now each of the figures $EABC$ and $DFBC$ are parallelograms (*by def. 36.*) and they are equal (*by 1. 36.*)

But the triangle ABC is half the parallelogram $EABC$, because the diagonal AB bisects it (*by 1. 34.*) Also the triangle DBC is half the parallelogram $DFBC$ (*by 1. 34.*) whence the triangle $ABC =$ triangle DBC (*by ax. 7.*) Q. E. D.

This proposition will be wanted in the demonstration of the second of the 6th book. By it also

also we are taught to find the contents of a triangle (B C D) because it is half the rectangle (A B C M) which stands on the same base, (B C) and has the same altitude (A B.)

PROPOSITION XXXVIII.

Triangles (A B C and D E F) standing on equal bases (B C and E F) and between the same parallels (B F and G D) are equal.

Draw B G parallel to A C and E H parallel to D F (by 1. 31.) Then the parallelograms G A B C and H D E F are equal (by E 1. 36.) wherefore the triangles A B C and D E F, which (by 1. 34.) are half the said parallelograms, are also equal (by ax. 7.)

By this proposition we are taught to divide a triangle (A B C) into any number of equal parts. Thus divide its base in the number of equal parts proposed, and to the points of division draw lines from the vertical angle (A) which will divide the given triangle into the proposed number of triangles, all equal because they stand on equal bases. This proposition will also be wanted in the 1st of the 6th book.

PROPOSITION XLVI.

Problem.

Upon a given line (A B) to describe a square (A B D E.)

From A draw A C perpendicular to A B (by prop. 11.) and make A D = A B (by prop. 3.) through

Through D, draw D E parallel to A B, and through B draw B E parallel to A D (*by prop. 31.*) then is the figure A B D E a square.

D E M O N S T R A T I O N.

For the figure A B D E is a parallelogram (*by construction and def. 36.*) whence (*by prop. 34.*) $B E = A D = A B$ (*by construction*) $= D E$ (*prop. 34.*)

Also because D E is parallel to A B, the angles $A + D =$ two right angles (*by prop. 29.*) but A is a right angle, wherefore D is also a right angle.

And because A and D are right angles, their opposite angles E and B are also right angles; (*by 1. 34.*) consequently (*by def. 30.*) the figure A B D E is a square. Q. E. D.

The construction of a square is very often wanted in practical mathematics, and will be made use of in many parts of the ensuing work.

Euclid's Elements.

BOOK III.

DEFINITIONS.

II.

A right line (A) is said to touch a circle when it meets the circumference, but does not cut it when produced.

A line touching a circle is called a Tangent.

VI.

*A segment (A B C) of a circle is a figure contained by a straight line * (A B) and that part of the circumference which it cuts off.*

VIII.

An angle (A C B) in a segment, is the angle formed by two right lines (A C and C B) drawn from any point (C) in the circumference

* Any part of the circumference of a circle is commonly called an arch: and the right line joining the extremities of an arch is called the chord of that arch. A *segment* is therefore better defined *a figure contained under an arch (A C B) and its chord A B.*

rence to the extremities of the chord (A B) of the segment.

PROPOSITION I.

Problem.

To find the center (C) of a given circle (A E B F.)

Draw within the given circle any right line (A B) bisect it in D (*by* 1. 10.) and draw D F perpendicular to A B and produce it to E. Lastly bisect E F in C, which is the center of the given circle.

DEMONSTRATION.

First I say the center (C) must fall somewhere in the line D F.

For if it does not, let the center be in any point, G, out of the line D F, and draw G A, G B and G D.

Then because G is the center, $GA = GB$ (*by* def. 15. B 1.) Also $AD = DB$ (*by* construction.) wherefore the two triangles A D G and B D G having three sides in one respectively equal to the three sides of the other, the angle $ADG = BDG$ (*by* 1. 8.) = a right angle (*by* def. 10. B 1.)

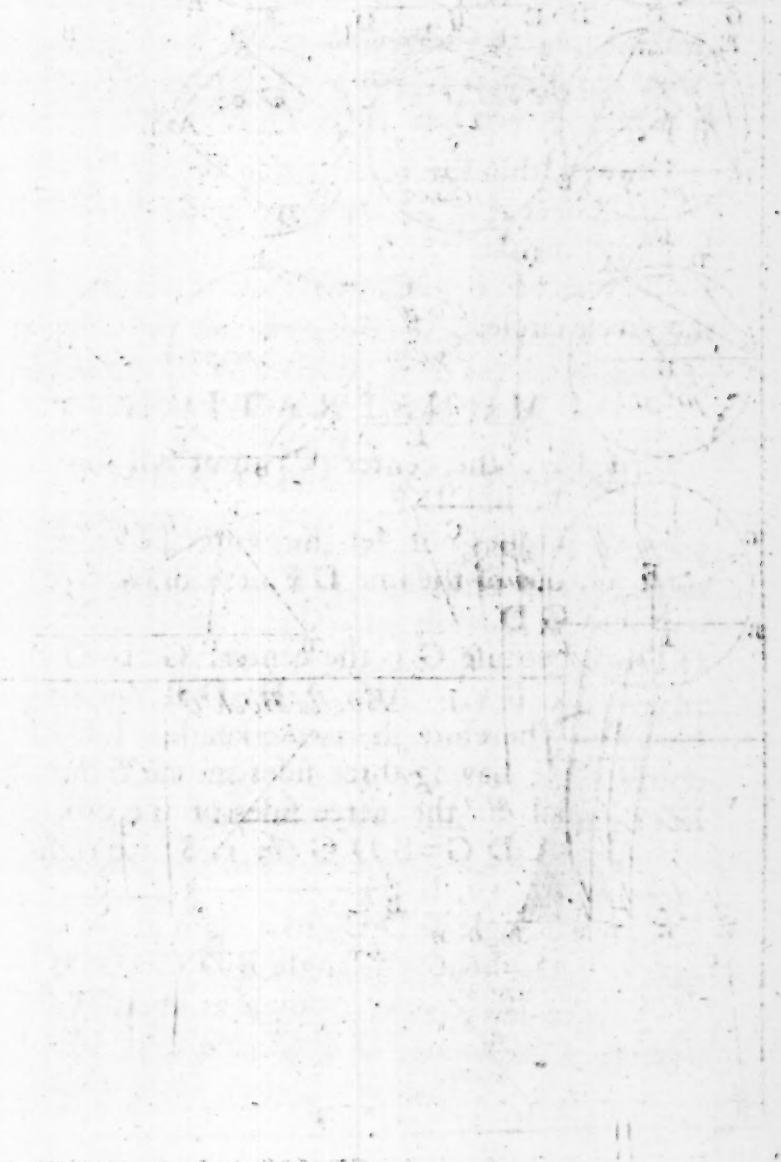
But the angle $BDG =$ to a right angle *by* construction) whence the angle $BDG = BDG$ the less to the greater, which is impossible. Wherefore the center of the circle is in F D.

and

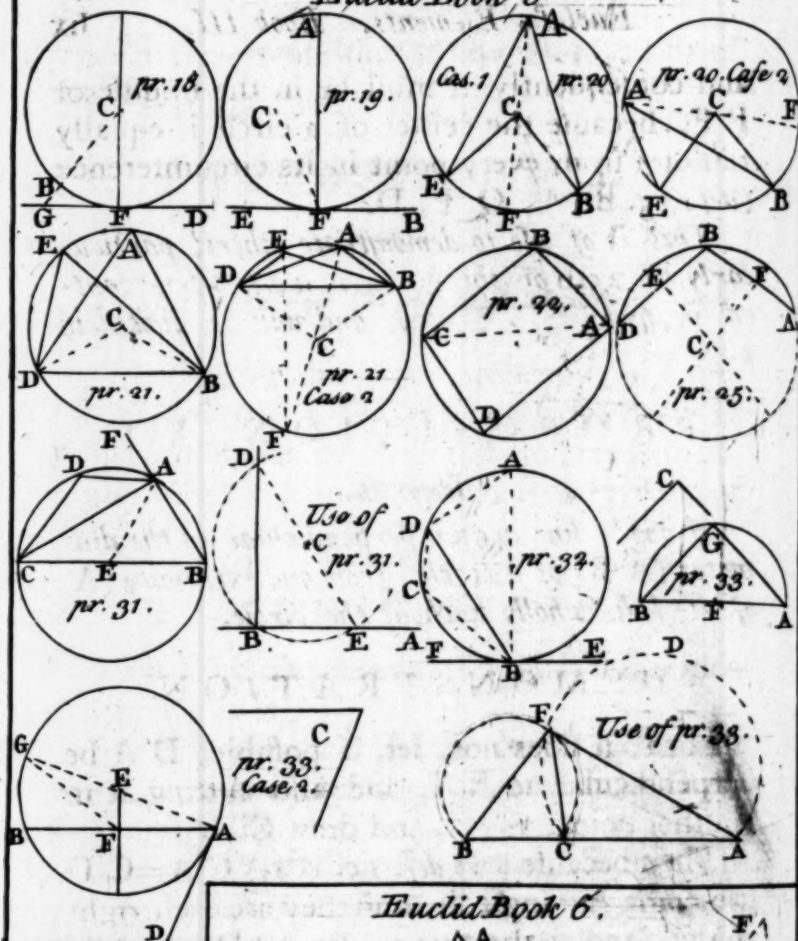
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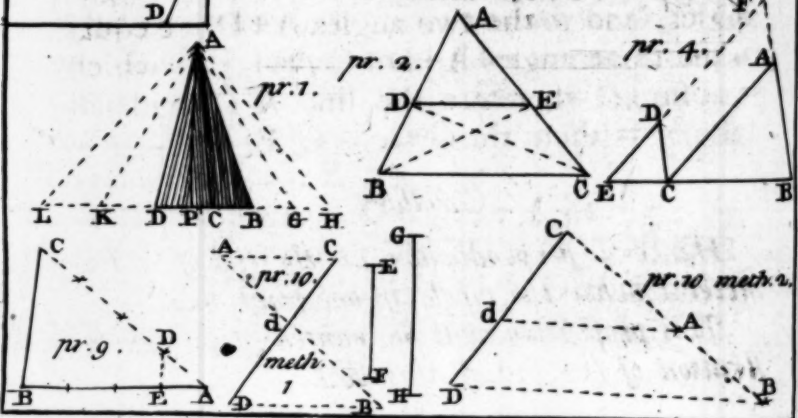
PROPOSITION



Euclid Book 3.



Euclid Book 6.



and consequently it must be in the middle of F E, because the center of a circle is equally remote from every point in its circumference (def. 15. B I.) Q. E. D.

This is of use to demonstrate others, particularly the 25th of this book: it is also often wanted in practical geometry, and will be quoted in page 223.

PROPOSITION XVI.

Theorem.

A right line drawn perpendicular to the diameter (A B) of a circle, from one extremity (A) of it, falls wholly without the circle.

DEMONSTRATION.

For if it does not, let, if possible, D A be perpendicular to B A, and also cutting it in another point, as (D) and draw C D.

Then because (by def. 15. B I.) $CA = CD$ the angle A = angle D, and they are each right angles, and so the two angles A + D are equal to the three angles A + D + C (by 1. 32.) which is absurd: wherefore the line A D must fall wholly without the circle. Q. E. D.

Corollary.

Hence a perpendicular at the end of a diameter touches the circle in one point only.

This proposition will be wanted in the demonstration of the 33d of this book.

PROPOSITION XVIII.

Theorem.

If a right line (D G) touches a circle, the line (C F) drawn from the center (C) to the point of contact (F) shall be perpendicular to the tangent (D G.)

DEMONSTRATION.

If it be denied that C F is perpendicular to F G, let if possible any other line, as C G, be drawn from C perpendicular to the tangent D G. Then because the triangle C G F is right angled at G; the angle F must be less than a right angle (by I. 32. cor. 3.) and C F must be greater than C G (by I. 19.) But C F = C B (by def. 15. B 1.) wherefore C B is greater than C G, a part than the whole: which is impossible; therefore F C is perpendicular to D G. Q. E. D.

This proposition serves to demonstrate the next and also will be quoted in pages 152 and 163.

PROPOSITION XIX.

Theorem.

If a right line (F A) be drawn perpendicular to a tangent of a circle (B E) from the point of contact (F); the center of the circle shall be in that line.

DEMON-

DEMONSTRATION.

For if the center is not in the line FA , let it be (if possible) in the point C and draw CF .

Then the angle CFE is a right angle (by 3. 18.) But AFE is also a right angle (by supposition) whence $CFE = AFE$ the less to the greater, which is impossible. In like manner it may be shewn that the center cannot be in any point out of the line AF , and consequently must be in it. Q. E. D.

This is wanted to demonstrate the 32d proposition of this book.

PROPOSITION XX.

Theorem.

The angle at the center of a circle is double to an angle at the circumference, which stands on the same arch.

DEMONSTRATION.

CASE I.

Let BCE be an angle, whose angular point C is at the center of the circle, and standing on the arch BE ; also let BAE be an angle, whose angular point A is in the circumference, and standing on the same arch BE ; also let the center C be within the an-

gle BAE : then I say the angle BCE is double the angle BAE .

Draw AC and produce it to F . Then because $BC = CA$, the angle $BAF = ABC$ (by I. 5.) whence angle $BAF + ABC = \text{twice } BAF$, also $BCF = BAF + ABC$ (by I. 32.) wherefore $BCF = \text{twice } BAF$: for the same reason FCE equal to twice FAE : consequently the whole angle BCE equal to twice the whole angle BAE .

CASE II.

When the center C is not within the angle BAE .

Through C draw CA , and produce it to F , then FCE is double FAE , also FCB (which is a part of FCE) is double FAB which is a part of FAE , wherefore the remaining angle BCE is double the remaining angle BAE . Q.E.D.

This is wanted to demonstrate the next proposition: it is also of great use in many parts of practical mathematics, particularly in the stereographic projection of the sphere in plano.

PROPOSITION XXI.

Theorem.

All angles (BAD and BED) in the same segment ($BAED$ vide def. 8.) are equal.

DEMON-

DEMONSTRATION.

CASE I.

If the segment (BAED) is greater than a semicircle. From the center (C) draw CB and CD, then the angle B A D is equal to half B C D (by 3. 20.) Also the angle B E D is equal to half B C D (by 3. 20.) wherefore B A D is equal to B E D (by ax. 7. B. 1.) Q. E. D.

CASE II.

If the segment (BAED) is not greater than a semicircle. Through the center (C) draw AC and produce it to F, also draw EF, BG and CD. Then (by the 1st case.) the angle B A F = B E F, also the angle F A D = F E D, wherefore (by ax. 2. B. 1.) the angle B A D (= B A F + F A D) = B E D (= B E F + F E D.) Q. E. D.

By this proposition we can find several stations situated at very unequal distances from the front of a building; from all which the said front shall appear of an equal width for it is a principle in optics, that objects which form equal angles at the eye appear equal; wherefore let AB be the ground plan of a wall, &c. take any distance AC in the compasses, and from A and B as centers, describe arcs cutting in C; from C with the same radius describe the segment ADEB, then if a man walk round the periphery of this circle, the length of the wall AB will

will every where appear equal, because its appearance will be measured by equal angles $A D B$, $A E B$, &c.

PROPOSITION XXII.

Theorem.

The opposite angles (B and D) of any quadrangle (ABCD) inscribed in a circle are equal to two right angles.

DEMONSTRATION.

The angle B is measured by half the arch ADC (by 3. 29.) also the angle D is measured by half the arch ABC.

But the arches $ADC + ABC$ is equal to the whole circumference, or 360° (vide def. 9. book 1.) Consequently, the angle $B + D$ is equal to half a circumference; or two right angles; or 180 degrees. Q. E. D.

This proposition will help to demonstrate the 31st and 32d of this book.

PROPOSITION XXV.

Problem.

An arch (ABD) of a circle being given to find its center (C).

To any point (B) in the given arch, draw lines (AB) and (BD) from its extremities (A and D)

Bisect

Bisect these lines BD and AB (by 1. 10.) in the points E and F : and (by 1. 11.) draw the perpendiculars EC and FC , cutting each other in the point C , which is the center of the circle whose arch was given.

DEMONSTRATION.

Since BD and BA are chords (by *def.* 6. book 3.) and since EC and FC are perpendiculars passing through their middles (E and F) the center of the circle must be in both these lines (by the demon. of 3. 1.) and consequently must be their intersection (C). Q.E.D.

This problem is of great use in many mathematical operations, and will be quoted in page 224.

PROPOSITION XXXI.

Theorem.

The angle (BAC) that is contained in a semicircle, is a right angle: the angle (ABC) contained in a greater segment, is an acute angle, and the angle (ADC) contained in a less segment than a semicircle is obtuse.

DEMONSTRATION.

Draw, from the center of the circle (E) the line EA , and produce BA to F ; then because $BE = EA$, the angle $EBA = \text{angle } BAE$ (by 1. 5.) also because $AE = EC$, the angle $EAC = ACE$; wherefore the two angles $ABC + ACB$ are equal

equal to the whole angle BAC . But the angle FAC is also equal to the two angles $ABC + ACB$ (by 1. 32.) wherefore (by ax. 1. book 1.) the angle $FAC = \text{angle } BAC$, and are both right angles (by def. 10. book 1) consequently the angles (BAC), in a semicircle, is a right line.

Again because the angle FAC is a right angle, and is also equal to both the angles $ABC + ACB$ (by 1. 32.) the angle ABC alone, is less than a right angle.

Again, because $ABCD$ is a quadrangle inscribed in a circle; the opposite angles $ABC + ADC$ is equal to two right angles (by 3. 22.) and it has been shewn that ABC alone, is less than one right angle; it follows that ADC is greater than a right angle.

This proposition shews us how to perform the 11th and 12th propositions of the first book, when the given point is situated near the end, or nearly opposite the end of the given line. Thus if it was required to raise a perpendicular to the line AB , from the point B , situated in the same, and near the end of it.

Set one point of the compasses in any point C , out of the given line AB , and with the distance CB describe a large portion of a circle cutting the given line in E . Through E and C draw ED cutting the periphery in D . Lastly, draw DB , which is the perpendicular sought. For it is evident the angle ABD is contained in the semicircle EBD , and therefore is a right angle.

If

If D is a point situated out of the given line AB , and nearly opposite its end, through which it is required to draw a perpendicular to AB , proceed thus:

Through D , draw any line at pleasure, ED , cutting the given line (AB) in E . Bisect (by 1. 10.) ED in C : at C as a center, with the radius CE or CD describe the semicircle DBE , cutting the given line AB in B . Lastly, draw DB , which shall be perpendicular to AB , by the last proposition.

There are many other uses to be made of this proposition, it will also be wanted to demonstrate prop. 33. of this book.

PROPOSITION XXXII.

Theorem.

If a right line (EF) touches a circle, and from the point of contact (B) a line (BD) be drawn cutting the circle, the angles made by this chord (BD) with the tangent (vide def. 2. and 6.) is equal to the angles contained in the alternate segments: viz. The angle DBF is equal to the angle in the segment BAD , and the angle DBE is equal to the angle in the segment DCB .

DEMONSTRATION

From the point of contact (B) draw BA perpendicular to EF , cutting the opposite part of the circumference in A , and draw AD .

Then

Then (by 3. 19.) BA passes through the center, and the angle BDA is a right angle (by 3. 31.) wherefore the angle $ABD + BAD$ is equal to a right angle (by 1. 32. corol. 3.) also the angle $ABD + DBF$ is equal to a right angle (by construct.) whence (by ax. 3. book 1.) $DBF = BAD$, which is the angle contained in the alternate segment; moreover, because $ABCD$ is a quadrangle in a circle, the angle $BAD + BCD$ is equal to two right angles (by 3. 22.) also the angle $DBF (=BAD) + DBE$ is equal to two right angles (by 1. 13.) whence $DBE = \text{angle } BCD$ (by ax. 3. book 1.) which is the angle in the alternate segment. Q.E.D.

This proposition will serve to demonstrate the next.

PROPOSITION XXXIII.

Problem.

Upon a given line (AB) to describe the segment (AGB) of a circle, that shall contain a given angle (C) (vide def. 6. and 8.)

CASE I.

If the given angle (C) is a right angle.

Bisect AB in F (by 1. 10.) and from F , as a center, describe the semicircle AGB , then is AGB a right angle (by 3. 31.)

CASE

CASE II.

If the angle (C) is not a right angle.

At the point A make the angle $BAD = C$ (by 1. 23.) also (by 1. 11.) draw AG perpendicular to AD, and bisect (by 1. 10.) AB in F, and draw FE perpendicular to AB, and cutting AG in E. Lastly, from E, as a center, with the radius EA, describe the arch AGB, which will pass through B, and form a segment (AGB) which will contain an angle equal to a given angle C.

DEMONSTRATION.

Draw EB, then in the triangles AFE and EFB, we have AF and FE with the angle AFE respectively equal to FB and FE, and the angle EFB (by construct.) wherefore (by 1. 4.) $EB = AE$, consequently the arch AGB must pass through B.

Again, because AD is perpendicular to the radius AE (by construct.) it is a tangent (by 3. 16.) and because AB is a line drawn within the circle from the point of contact (A); the angle DAB (equal to the given angle C) is equal to the angle contained in the alternate segment (by 3. 32.) Q. E. D.

By the help of this proposition we may find a point from which two very unequal parts of the same line shall appear equal. Thus, suppose AB to be a line divided into two unequal

equal parts in the point C, and it is required to find a point F from whence those parts shall appear equal; on one part A C describe (*by this prob.*) a segment A D F C that shall contain any angle at pleasure; also on the part C B describe a segment C F E B that shall contain an angle equal to the angle contained in the segment A D F C: then the point F, where these arcs cut each, will be the point sought. For at F, the appearance of C B is measured by the angle C F B, and the appearance of A C is measured by the angle A F C; which angles (*by this construction* are equal).

This proposition will be quoted page 290.

Euclid's Elements.

B O O K V.

*I*N the following demonstrations I shall make use of letters to denote the quantities, whose proportional properties are to be investigated; and when two or more quantities, connected with the sign + or —, have a line drawn over them, they are to be esteemed as that one quantity which would arise by adding and subtracting the several quantities as their signs indicate.

Thus $\overline{A+B+C}$ is to be understood as that one quantity, arising from adding the quantities represented by A, B, and C, together: also $\overline{A+B-C}$ denotes the single quantity, which results when C is subtracted from the sum of A and B.

DEFINITIONS.

I.

A lesser magnitude is said to be a part of a greater, when the lesser measures the greater: or when the lesser is contained in the greater a certain number of times.

Thus

Thus 4 is a part of 12, being contained 3 times therein: also a foot is a part of a yard because it is contained three times in it.

II.

A greater magnitude is said to be a multiple of a lesser, when the greater is measured by the less.

Thus, 12 is a multiple of 3, a yard is a multiply of a foot, and a shilling is a multiple of a groat.

III.

Ratio is the mutual relation two magnitudes of the same species have to each other in respect of quantity.

When two magnitudes are compared together, the former is called the ANTECEDENT and the latter the CONSEQUENT: and the RATIO or PROPORTION between them, is found by dividing the number of equal parts contained in the antecedent, by the number of like parts contained in the consequent. Thus $\frac{20}{1}$ is the RATIO or PROPORTION which a pound has to a guinea: because the ANTECEDENT contains 20 shillings and the CONSEQUENT 21.

Number is the general medium through which all our ideas of magnitude or quantity are conveyed: we have but a very confused notion of the relation which two magnitudes have to each other,
till

'till we either actually divide them, or conceive them to be divided, into parts of equal quantity : and we estimate their ratio by the numbers of such equal parts contained in the two magnitudes.

Thus if we compare two bodies by their weight, we put them in a pair of scales, and find how many pounds or ounces are contained in each ; and the ratio between them, we estimate by the proportion of these numbers to each other.

If we estimate them by their bulk, we compute how many spaces equal to a cubical foot or inch, &c. is contained in each, and the numbers expressing these, give us an idea of the relation their bulks have to each other.

IV.

Quantities are said to be in some ratio, or other, when they are capable of being multiplied so as to exceed each other.

From hence it appears that heterogeneous quantities cannot have any ratio to each other : thus a yard and pound have no ratio, because neither can be multiplied 'till it exceed the other, for the same reason a line cannot be compared with a surface, nor a surface with a solid.

V.

† Four magnitudes are said to be in the same
f ratio

† This definition though just, and calculated to introduce the most elegant and unexceptionable demonstrations of the properties of proportional quantities, nevertheless, does not at first sight coincide with that idea of proportionality, which every man feels in himself : and because
wherever

ratio to the first, as to the second, as the third

to

wherever Euclid proves four quantities to be proportional, it is from having this property; his whole 5th book has an air of ambiguity, which is apt to discourage learners from perusing it: and by this means the most elegant, and useful book of his Elements is commonly the least understood; for this reason, every editor of Euclid has endeavoured to remove this stumbling block, either by substituted some other definition in its room, or lessening its obscurity by explications and comments. But none seem to me to have succeeded so well as Dr. Saunderson in the 7th book of his Algebra, from whom I shall borrow the heads of this Note.

The most common and natural idea of the proportionality of four quantities, from whence we may perceive children and market-folks, self taught, derive their little computations is this:

Four quantities (A, B, C, and D) are proportional, if the first (A) is the same part or parts of the second (B) that the third (C) is of the fourth (D).—Thus if A is $\frac{1}{2}$ of B and C is $\frac{1}{2}$ of D, then is A the same part of B, that C is of D: and the four quantities are proportionable.

Again if $A = \frac{1}{2} B$ and $C = \frac{1}{2} D$; or if $A = \frac{2}{3} B$ or $2 B$ and $C = \frac{2}{3} D$ or $2 D$, or if $A = \frac{1}{3} B$ and $C = \frac{1}{3} D$; in all these cases A is the same parts of B that C is of D, and therefore A has the same proportion to B that C has to D.

Now, though this is an infallible sign of proportionality, and will answer every purpose where numbers only are concerned, yet to be universal we must have a more extensive criterion. For instance, it may be demonstrated, that the diagonal of any square is incommensurable to its side, or that no part, or parts of the diagonal can be taken that shall equal the side of the square. Yet it may also be demonstrated, that the side (A) of any square has the same proportion to its diagonal (B) as the side (C) of any other square has to its diagonal (D); yet A is not the same part or parts of B that C is of D, because A is no part

to the fourth, when no equimultiples what-

f 2

soever

part or parts of B nor C of D. For if the side of a square is 1, its diagonal will be equal to the square root of 2, ≈ 1.414 or more nearly 1.4142 or still more nearly 1.41421 whence it follows, that if the side is divided into ten equal parts, the diagonal will contain more than fourteen but less than fifteen of those parts. If the side is divided into a hundred equal parts, the diagonal will contain more than a hundred and forty-one, but not a hundred and forty-two of those parts, &c.

In order to establish a more general definition, let us assume the following principle, which accords so nearly with the natural idea of proportionality, that every body will easily subscribe to its truth.

PROPOSITION.

If there are four quantities A, B, C and D, whereof A has the same proportion to B that C has to D; it will then be impossible for A to be greater than any part or parts of B, but C must also be greater than a like part or parts of D. Or if A is equal to any part or parts of B, C will also be equal to a like part or parts of D; or if A is less than any part or parts of B, C will also be less than the like part or parts of D. Thus if A is greater than, equal to, or less than $\frac{1}{10}$ of B, C will also be greater than, equal to, or less than $\frac{1}{10}$ of D.

This principle is so immediately derived from the fore-mentioned natural idea of proportionality, that no demonstration can render it more evident; and if we can demonstrate the converse, we shall have an idea of proportionality as extensive as the subject itself; now the converse of the above principle is this:

PROPOSITION.

If there are four quantities A, B, C, D, so related together that A cannot possibly be greater than, equal to, or less than, any part or parts of B; but that C will also be greater than, equal to, or less than the like part or parts of D, then I say that A must have the same proportion to B, that C has to D.

DEMON-

soever of the first and third can possibly be taken,

DEMONSTRATION.

For if it be denied that C has the same proportion to D that A has to B, let any other quantity G have the same proportion to D that A has to B; that is A is to B, so is G to D; also suppose that if D was divided into ten equal parts, that G is greater than fourteen but less than fifteen of these parts, then since

G is greater than $\frac{14}{10}$ of D, and less than $\frac{15}{10}$ of D.

A is greater than $\frac{14}{10}$ of B and less than $\frac{15}{10}$ of B. } *by the last proposition.*

But since A is greater than $\frac{14}{10}$ of B and less than $\frac{15}{10}$. C must be greater than $\frac{14}{10}$ of D and less than $\frac{15}{10}$. } *by supposition.*

But G is also greater than $\frac{14}{10}$ of D and less than $\frac{15}{10}$ (as above.)

We have now proved, that if G is between $\frac{14}{10}$ and $\frac{15}{10}$ of D, C will also be contained between the same limits, and therefore the difference between C and G must be less than $\frac{1}{10}$ of D.

This was upon supposition that D was divided into ten equal parts, but if D had been divided into a hundred, a thousand, ten thousand, &c. equal parts, the conclusion would have been that C and G differed from each other, by a quantity less than the hundredth, thousandth, ten thousandth, &c. part of D; consequently the difference between C and G must be less than any part of D however small, or in other words C and G must be equal.

Since then C is equal to G, and as A is to B as G is to D (*by supposition*) it follows that A is to B, so is C to D. Q. E. D.

It is highly probable that Euclid would have made this proposition his criterion of proportionality, as being more natural and agreeable to the general idea of it; but had he done he must in demonstrating the properties and affections of proportionate magnitudes, have supposed them

ken, but what shall be both respectively greater,

f 3

or

them to be divided into such and such parts, instead of taking such and such multiples of them; which would have been less elegant as the use of whole numbers is in all cases more easy and natural than that of fractions, and the multiplication of quantities is more simple and easy to the conception than the resolution of them into parts.

For these reasons; it was necessary to advance the idea of proportionality a step farther, by substituting multiples instead of all aliquot parts: and instead of taking the last proposition for a definition of proportional quantities it is better to define them by that property, which shall be demonstrated in the following

PROPOSITION.

If there be four quantities A, B, C, D, whereof $E \times A$ and $E \times C$ are any equimultiples of the first and third: and $F \times B$ and $F \times D$ are any other equimultiples of the second and fourth; and if these quantities are of such a nature, that $E \times A$ cannot be greater than, equal to, or less than $F \times B$, but that $E \times C$ must necessarily be greater than, equal to, or less than $F \times D$, be these multipliers E and F any numbers whatsoever. I say then that A has the same proportion to B that C hath to D.

DEMONSTRATION.

If it be denied that A is to B as C is to D. Let A be to B as any other quantity G is to D; that is, let $A : B :: G : D$. Also suppose D is divided into ten parts, and that G is greater than fourteen and less than fifteen of those parts, then must A also be greater than $\frac{14}{10}$ and less than $\frac{15}{10}$ than of B (by supposition.)

Now since A taken once is greater than $\frac{14}{10}$ of B, it is evident that A taken ten times is greater than $\frac{14}{10}$ of B. But a hundred and forty tenth parts is equal to fourteen wholes, wherefore $\frac{14}{10}$ of B is equal to fourteen B:

whence

or both equal to, or both respectively less than any

whence (A taken ten times or) ten A is greater than fourteen B.

In like manner since A taken once is less than $\frac{1}{10}$ of B, (A taken ten times, or) ten A must be less than (1 or) fifteen B; but because ten times A is greater than fourteen and less than fifteen times B. Ten times C must be greater than fourteen and less than fifteen times D (by our hypothesis) and once C must be greater than a tenth part of fourteen and less than a tenth part of fifteen times D; but a tenth part of fourteen D, or fifteen D is $\frac{1}{10}$ of D or $\frac{1}{2}$ of D, consequently C is greater than $\frac{1}{10}$ and less than $\frac{1}{2}$ of D, but by supposition C is also greater than $\frac{1}{10}$ and less than $\frac{1}{2}$ of D, therefore the difference between C and G must be less than a tenth of D. but this was upon supposition that D was divided into ten equal parts, for had D been supposed to be divided into a hundred, a thousand, ten thousand, &c. parts, the difference between C and G would have been less than one of those parts, wherefore C and G have less difference than any part of D whatsoever; or in other words $C=G$, and since it was agreed that A is to B as G is to D, it follows that A is to B as C is to D. Q. E. D.

We are now arrived to the criterion of proportionality expressed in the 5th definition: to wit,

Four quantities A, B, C, D, are proportional where any equimultiples whatsoever, ($E \times A$ and $E \times C$) of the first and third are both greater than, both equal to, or both less than any other equimultiples whatsoever, ($F \times B$ and $F \times D$) of the second and fourth respectively; that is,

If $E \times A$ is greater than $F \times B$, $E \times C$ is also greater than $F \times D$, or if $E \times A$ is equal to $F \times B$, $E \times C$ is also equal to $F \times D$, or if $E \times A$ is less than $F \times B$, $E \times C$ is also less than $F \times D$.

The reader may, perhaps, think this note tedious, but he may assure himself its importance is equal to its length and much greater than its difficulty; as he will perceive at the second or third reading.

any other equimultiples, that can possibly be taken of the second and fourth; to wit, if A, B, C, D, are four quantities, and it can be proved that no equimultiples (as $E \times A$, and $E \times C$) of the first and third can possibly be taken, but what must necessarily be both respectively greater than, both equal to, or both respectively less than any other equimultiples ($F \times B$ and $F \times D$) that can possibly be taken of the second and fourth, than A has the same ratio to B that C has to D, or $A : B :: C : D$.

VI.

Magnitudes that have the same ratio are called proportionals.

Thus if A has the same ratio (*vide note to def. 3.*) to B that C has to D, those quantities are proportionals, and are wrote $A : B :: C : D$, and they are read thus:

A is to B so is C to D.

When four quantities are proportionals, the first and third terms are called antecedents and the second and fourth are called consequents.

XII.

In proportional quantities the antecedents are called homologous to each other, and the consequents are also said to be homologous to each other.

XIII.

Alternate proportion is, when four quantities, being proportionable, the first to the second, as the third to the fourth; it is concluded that the first is to the third as the second is to the fourth. Thus if $A : B :: C : D$, it is concluded, that $A : C :: B : D$, the truth of which conclusion will be demonstrated in prop. 16.

XIV.

Inverse proposition is, when four quantities being proportionable, the first to the second, as the third to the fourth; it is concluded that the second is to the first as the fourth is to the third †. Thus if $A : B :: C : D$ it is concluded that $B : A :: D : C$.

XV.

Composition of proposition is, when four quantities being proportional, the first to the second,

† The truth of this is so self-evident, that Euclid does not attempt a demonstration although it is used to demonstrate prop. 19. but its truth may be demonstrated thus:

Let $3A$ and $3C$ be any equimultiples of A and C , and let $2B$ and $2D$ be any other equimultiples of B and D then since $A : B :: C : D$, if $3A$ is greater than, equal to, or less than $2B$; $3C$ will be greater than, equal to, or less than $2D$ (*by def. 5.*) or in other words, if $2B$ is less than, equal to, or greater than $3A$; $2D$ must be less than, equal to, or greater than $3C$, consequently (*by def. 5.*) $B : A :: D : C$. Q. E. D.

second, as the third to the fourth; it is concluded, that the sum of the first and second is to the second, as the sum of the third and fourth is to the fourth: that is if $A : B :: C : D$. We conclude $A+B : B :: C+D : D$.

The truth of this conclusion will be demonstrated in prop. 18.

XVI.

Division of proportion is, when four quantities are proportionable, the first to the second as the third is to the fourth, and it is concluded that the excess of the first above the second, is to the second, as the excess of the third above the fourth is to the fourth. Thus if $A : B :: C : D$, it is concluded that $A-B : B :: C-D : D$.

The truth of this conclusion will be demonstrated in prop. 17.

XVII.

Conversion of proportion is, when four quantities are proportional, the first to the second as the third is to the fourth, and it is concluded that the first is to the excess of the first above the second, as the third is to the excess of the third above the fourth. Thus if $A : B :: C : D$ it is concluded that $A : A-B :: C : C-D$.

The truth of this conclusion will be shewn in the corollary to prop. 19.

If

A X I O M.

If two quantities A and C are equal, any multiple part or parts of A, will be equal to a like multiple, part or parts of C. Thus if $A=B$; $3 A$ is $= 3 B$; $\frac{1}{4} A = \frac{1}{4} B$ or $\frac{1}{4} A = \frac{1}{4} B$.

But if A is less than B, then is any multiple, part or parts of A, less than a like multiple, part, or parts of B. Thus if A is less than B; then $3 A$, or $\frac{1}{4} A$, or $\frac{1}{4} A$ are respectively less than $3 B$, or $\frac{1}{4} B$, or $\frac{1}{4} B$.

But if A is greater than B, then is any multiple, part, or parts of A greater than a like multiple, part or parts of B.

P R O P O S I T I O N VII.

Theorem.

Equal quantities have the same proportion to the same quantity: also the same quantity has the same proportion to equal quantities.

Thus if $A=B$, and C is any other quantity, A will have the same proportion to C that B has; also C will have the same proportion to A that it has to B.

This is as evident as any axiom in the Elements.

This theorem will be wanted in the demonstration of prop. 2. book 6.

PROPO-

PROPOSITION IX.

Theorem.

If two quantities A and B have the same proportion to a third C; or if any quantity C has the same proportion to two other quantities A and B in either of these cases the quantities A and B are equal.

This is self-evident, and made an axiom by the best modern writers on the Elements of Geometry, Mr. Thomas Simpson, and Mr. William Payne.

By this proposition we often prove that two quantities are equal to each other, by proving they have the same proportion to some other, as is exemplified in page 95.

PROPOSITION XI.

Theorem.

If two ratios be the same with a third, they must be the same with one another; as if the ratio of A to a, and the ratio of C to c, be both the same with the ratio of B to b; then the ratio of A to a will be the same as the ratio of C to c; or thus if

$A : a :: B : b$ and

$B : b :: C : c$ I say then that

$A : a :: C : c.$

DEMONSTRATION.

For let 3 A, 3 B and 3 C be any equimultiples of the antecedents A, B and C; also let 2 a, 2 b and 2 c be any other equimultiples of the consequents a, b and c; also suppose 3 A is greater than 2 a, then because

A

$A : a :: B : b$; $3 B$ must be greater than $2 b$: also because $B : b :: C : c$; $3 C$ must be greater than $2 c$, wherefore in like manner we might prove that if $3 A$ was equal to $2 a$, $3 C$ would also be equal to $2 c$, or if $3 A$ was less than $2 a$, $3 C$ would also be less than $2 c$, wherefore A, a, C and c are four quantities so related to each other that if any equimultiples (as $3 A$ and $3 C$) are taken of the first (A) and third (C) and any other equimultiples (as $2 a$ and $2 c$) are taken of the second and fourth, it is impossible for $3 A$ to be greater than, equal to, or less than $2 a$, but that $3 B$ will also be greater than, equal to, or less than $2 b$; whence (*by def. 5.*) $A : a :: C : c$.
Q. E. D.

This proposition is often wanted in demonstrating the principles of perspective, as will be seen in pages 44, 53, 54, 95, 117, 288, &c.

PROPOSITION XIV.

Theorem.

If four quantities of the same kind are proportional, the first to the second as the third to the fourth; the second quantity will be greater than, equal to, or less than the fourth, according as the first quantity is greater than, equal to, or less than the third.

DEMONSTRATION.

For let $A : B :: C : D$; also suppose $A = C$, then I say $B = D$. For since $A = C$,
 and

and since B and D is the same multiples part or parts of A that D is of C (*Vide* page 74.) it follows (by axiom B 5.) that D is equal to B.

In the same manner it may be proved, that if A is greater or less than C; B is also greater or less than D. Q. E. D.

The use of this is to demonstrate the 16th proposition.

PROPOSITION XV.

Theorem.

Quantities are in the same proportion as their equimultiples. Thus if A and B are any two quantities whereof 3 A and 3 B are any equimultiples whatsoever; then I say

$$A : B :: 3 A : 3 B.$$

DEMONSTRATION.

For let 2 A and 6 A, be any equimultiples of the first and third (A and 3 A) also let 3 B and 9 B be any other equimultiples of 2d and 4th quantities (B and 3 B.) Then it is evident that if 2 A is greater than 3 B; 6 A must also be greater than 9 B, or if 2 A is equal to 3 B; 6 A must also be equal to 9 B, or if 2 A is less than 3 B; 6 A must also be less than 9 B: wherefore by def. 5 the quantities A, B, 3 A, 3 B are proportional, and $A : B :: 3 A : 3 B$. Q. E. D.

The theorem is wanted to demonstrate the next.

PROPO-

PROPOSITION XVI.

Theorem.

If four quantities of the same kind are proportional, they will be proportionable when taken alternately: or the two antecedents have the same ratio to each other as the two consequents (Vide def. 13.) That is if $A : B :: C : D$.

I say then $A : C :: B : D$.

DEMONSTRATION.

Let 3 A and 3 B be any equimultiples of the 1st and 2d quantities; also let 2 C and 2 D be any other equimultiples of the 3d and 4th. Then $A : B :: 3 A : 3 B$ (by 5. 15.) Also $A : B :: C : D$ (by supposition.) Wherefore $3 A : 3 B :: C : D$ (by 5. 11.) But $C : D :: 2 C : 2 D$ (by 5. 15.) Consequently $3 A : 3 B :: 2 C : 2 D$ (by 5. 11.)

Wherefore (by 5. 14.) it is impossible for 3 B to be greater than, equal to, or less than 2 C, unless 3 B is also greater than, equal to, or less than 2 D, wherefore (by def. 5. B 5.) $A : C :: B : D$. Q. E. D.

This proposition will be quoted in page 53.

PROPOSITION XVII.

Theorem.

If four quantities A, B, C, D (whereof A is greater than B, and C is greater than D) are

are proportional, A to B as C is to D: I say then that

$$\overline{A-B} : B :: \overline{C-D} : D,$$

which is called proportion by division (*vide* def. 16. book 5.)

DEMONSTRATION.

Let $\overline{3A-3B}$ and $\overline{3C-3D}$ be any equimultiples of the terms $\overline{A-B}$ and $\overline{C-D}$.

Also let $2B$ and $2D$ be any other equimultiples of the terms B and D .

Also suppose $\overline{3A-3B}$ is greater than $2B$, add $3B$ to both these quantities, and we have $\overline{3A-3B+3B}$ is greater than $2B+3B$, that is, $3A$ is greater than $5B$; and because $A : B :: C : D$, and $3A$ is greater than $5B$, it follows, that $3C$ is greater than $5D$ (*by* def. 5. book 5.) take $3D$ from both these quantities, and we have $\overline{3C-3D}$ greater than $5D-3D$, that is $\overline{3C-3D}$ is greater than $2D$, whence we find, that if $\overline{3A-3B}$ is greater than $2B$, $\overline{3C-3D}$ is also greater than $2D$. And in the same manner it may be demonstrated, that if $\overline{3A-3B}$ is equal to, or less than $2B$; $\overline{3C-3D}$ will also be equal to, or less than $2D$.

Seeing then we have 4 quantities $\overline{A-B}$, B , $\overline{C-D}$, D , whereof $\overline{3A-3B}$ and $\overline{3C-3D}$ represent any equimultiples of the 1st and 3d; and $2B$ and $2D$ are any other equimultiples of the 2d and 4th; and since we have demonstrated, that

Ixxxviii *Euclid's Elements. Book V.*

that if $\overline{3A-3B}$ is greater than, equal to, or less than $2B$; $\overline{3C-3D}$ will also be greater than, equal to, or less than $2D$. It follows from (*def. 5. book 5.*) that these four quantities are proportional; *viz.* $\overline{A-B} : B :: \overline{C-D} : D$. Q. E. D.

This is a very useful proposition in most mathematical enquiries, but it is not quoted in the following work.

PROPOSITION XVIII.

Theorem.

If four quantities A, B, C, D are proportional, $A : B :: C : D$; I say then that

$$A+B : B :: C+D : D,$$

*which is called proportion by composition (vide *def. 15. book 5.*)*

DEMONSTRATION.

If it be denied that $A+B : B :: C+D : D$, let it be allowed, that $A+B$ is to B as $C+D$ is to some other quantity greater or less than D , and call this quantity E . That is, let $A+B : B :: C+D : E$.

Now if E is supposed greater than D let $\overline{C-E}$ be added to both these quantities, and we shall have $\overline{E+C-E}$ greater than $\overline{D+C-E}$, that is, C is greater than $\overline{C+D-E}$. This being observed, since $\overline{A+B} : B :: \overline{C+D} : E$ (by supposition) we have (*by 5. 17.*) $\overline{A+B-B} : B :: \overline{C+D-E} : \overline{E}$; that is, $A : B :: \overline{C+D-E} : E$.

But

But $A : B :: C : D$ (by the *hypothesis*) therefore as $C : D :: \overline{C+D-E} : E$ (by 5. 11.) But it has been proved that the first term C is greater than the third $\overline{C+D-E}$, wherefore the second term (D) is greater than the fourth term E (by 5. 14.)

Whence we gather, that if $\overline{A+B}$ is to B as $\overline{C+D}$ is to any other quantity greater than D ; that such quantity must be less than D , which is impossible, consequently it is impossible for $\overline{A+B}$ to be to B as $\overline{C+D}$ is to any quantity greater than D .

And in like manner it may be demonstrated that it is impossible for $\overline{A+B}$ to be to B as $\overline{C+D}$ is to any quantity less than D .

Wherefore $\overline{A+B}$ must be to B , as $\overline{C+D}$ is to D . Q. E. D.

This proposition is as useful as the former, and is quoted in pages 94 and 117.

PROPOSITION XIX.

If from any two quantities A and B, there are respectively subtracted any other two quantities C and D, that are in the same proportion; I say that the remainders $\overline{A-C}$ and $\overline{B-D}$ will be in the same proportion; that is $\overline{A-C}$ will be to $\overline{B-D}$ as A is to B ; or as C is to D .

DEMONSTRATION.

Since $A : B :: C : D$ (by *supposition*)
 $A : C :: B : D$ (by *alternation* 5. 16.) and

g

$\overline{A-C}$

$\overline{A-C} : C :: \overline{B-D} : D$ (by division 5. 17.) whence
 $\overline{A-C} : \overline{B-D} :: C : D :: A : B$. Q. E. D.

S C H O L I U M

Theorem.

If four quantities A, B, C, D are proportionable, A to B as C to D; I say then that $A : \overline{A-B} :: C : \overline{C-D}$, which is called conversion of proportion (vide def. 17. book 5.)

For since $A : B :: C : D$ (by supposition) we have $\overline{A-B} : B :: \overline{C-D} : D$ (by division 5. 17.) whence $B : \overline{A-B} :: D : \overline{C-D}$ (by inversion, vide def. 14.) and $\overline{B+A-B} : \overline{A-B} :: \overline{D+C-D} : \overline{C-D}$ (by composition 5. 18.) that is $A : \overline{A-B} :: C : \overline{C-D}$. Q. E. D.

Neither this proposition nor that contained in the last scholium will be wanted in the ensuing work; yet they are such useful affections of proportionable quantities as I could not but wish my reader to be acquainted with. There is so much ingenuity in the doctrine of proportion, that the more the mind becomes acquainted, the more it will be enamoured with it; and its investigation will afford the inquisitive as much entertainment, as its application affords advantages to all ranks and degrees of mankind.

The slothful are apt to repine because jewels are not to be had without digging for; and that the tillage must always precede the harvest: many of this cast would like to be masters of proportion if all its intricacies could be discovered

vered at a glance. But such will never be mathematicians or artists: they spend their whole lives in wishing for what the industrious acquire in a short space; and although, to the former, the doctrine of proportion may seem a most intricate mystery, the researches of the latter will assuredly be crowned with success; its difficulty will grow continually less, its abstruseness will vanish, and the pains it has cost, will serve to enhance the pleasure of the acquisition.

Euclid's

Euclid's Elements.

B O O K VI.

DEFINITION I.

SIMILAR right lined figures are those which have their several angles equal, each to each, and the sides about those angles proportional.

IV.

The altitude of a figure is a straight line, drawn from its vertex, perpendicular to, and terminating in, its base.

PROPOSITION I.

Theorem.

Triangles having the same altitude, are to each other as their bases.

Let ABC and ACD be two triangles, having the same altitude (viz. the line AP drawn

drawn perpendicular to, and terminated, by B D) and standing on the bases B C and C D, then I say, the triangle A B C : triangle A C D :: B C : C D.

DEMONSTRATION.

Produce the base line B D, and take D K and K L, &c. any number of times, each equal to C D, and also take B G and G H, any number of times, each equal to B C, draw A H, A G, A K, and K L.

Now because B C, B G, G H, &c. are equal, the triangles (*by* 1. 38.) standing on those parts are equal, wherefore the triangle A H C (which is the sum of all the others,) is the same multiple of the triangle A B C, that C H is of B C.

For the same reason the triangle A C L is the same multiple of the triangle A C D.

Now if the base H C is = C L, the triangle A H C = A C L (*by* 1. 38.); or if H C is greater or less than C L; the triangle A H C is also greater or less than A C L.

We have then these four quantities, viz. the base B C, the base C D, the triangle A B C and the triangle A C D; and it is proved that any equimultiples of the first and third, are both greater than, both equal to, or both less than any equimultiples that can be taken of the second and fourth, wherefore B C : C D :: triangle A B C : triangle A C D. Q. E. D.

PROPOSITION II.

Theorem.

If a right line (DE) be drawn parallel to one side (BC) of a triangle (ABC) it will cut the other two sides (AB and AC) proportionably, that is $BD : DA :: CE : EA$.

DEMONSTRATION.

Draw BE and DC. Then because the triangles BDE and CDE stand on the same base (DE) and are between the same parallels DE and BC, they are equal (by 1. 37.). Also the triangle BDE : triangle ADE :: triangle CDE : triangle ADE (by 5. 7.) But triangle BDE : triangle ADE :: BD : DA (by 6. 1.) whence $BD : DA ::$ triangle CDE : triangle ADE (by 5. 11.) But triangle CDE : triangle ADE :: CE : EA (by 6. 1.) whence $BD : DA :: CE : EA$. Q. E. D.

This proposition is truly valuable for its manifold uses; it serves to demonstrate the 4th and 9th following.

PROPOSITION IV.

Theorem.

If two triangles are equiangular, the sides about the equal angles are proportional: also those sides which are opposite to the equal angles, are homologous sides; that is, are the antecedents or consequents of the ratio's.

DEMON-

DEMONSTRATION.

Let the ABC and DCE be two triangles, having the angles A , B , and C in the one, respectively equal to the angles D , C , and E , in the other: also let the triangle DCE be so placed; that CE may be in the continuation of BC and contiguous to it.

Then because BE is a straight line, and the angle $ACB = E$, AC and ED will be parallel (*by 1. 28.*) also because the angle $DCB =$ angle B ; CD is parallel to AB .

Produce BA and ED till they cut each other in F ; then is the figure $ACDF$ a parallelogram (*vide def. 36. book 1.*)

Now because AC is parallel to FE . $BA : AF :: BC : CE$ (*by 6. 2.*) but $AF = CD$ (*by 1. 34.*) whence $BA : CD :: BC : CE$ (*by 5. 7.*) and $BA : BC :: CD : CE$ (*by 5. 16.*) again because BF is parallel to CD , $BC : CE :: FD : DE$ (*by 6. 2.*) but $FD = AC$ (*by 1. 34.*) whence $BC : CE :: AC : DE$ (*by 5. 7.*) and $BC : AC :: CE : DE$.
Q. E. D.

The various uses of this proposition would take a volume to describe: we shall have occasion for it in pages 31. 44. 53. 54. 94. 113. 114. and 115 of the following work.

PROPOSITION IX.

Problem.

From a given straight line (A B) to cut off any part (vide book 5. def. 1.) required.

From A, draw A C, making any angle with A B: and take any small opening (A D) in the compasses and lay it on A C, from A as often as the part to be cut off from A B is to be contained therein (suppose four times).

Draw B C and through D draw D E parallel to B C.

Then E A is the same part of A B that A D is of A C, that is if A C is four times, A D; A B is also four times A E.

DEMONSTRATION.

For because D E is parallel to B C, the angle B = angle E and the angle C = angle D (by 1. 29.) wherefore the two triangles A B C and A E D are equiangular, and $A D : A E :: A C : A B$ (by 6. 4.) whence $A D : A C :: A E : A B$ (by 5. 16.) Q. E. D.

The utility of this problem is universally known: it will be wanted in many parts of this treatise.

PRO-

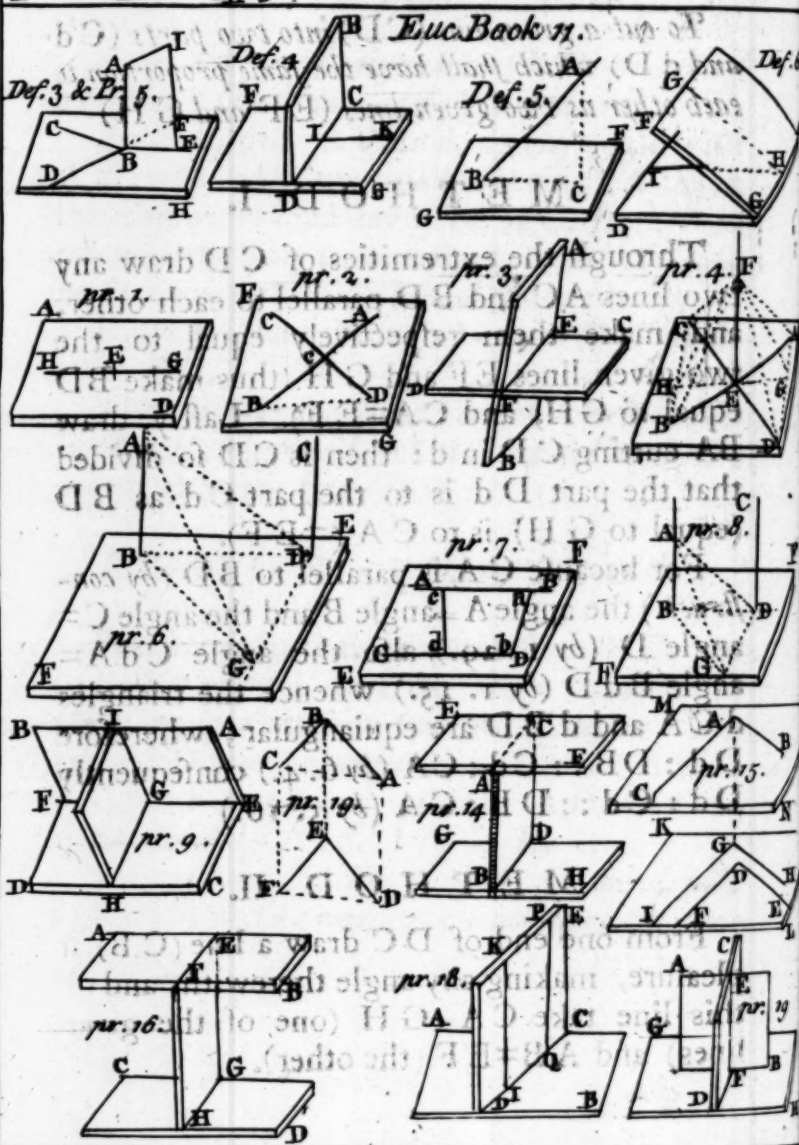
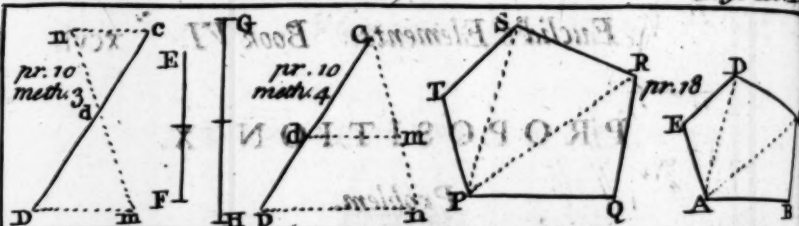
PROPOSITION IX

Problem

From a given straight line AB to cut off any part (side book 2. def. 1.) required.
From A , draw AC making any angle with AB : and take any small opening AD in the compasses and lay it on AC from A as often as the part to be cut off from AB is to be contained therein (suppose four times). Draw BC and through D draw DE parallel to BC .
Then EA is the same part of AB that AD is of AC : that is if AC is four times AD , AB is also four times EA .

DEMONSTRATION

For because DE is parallel to BC , the angle $B =$ angle E and the angle $C =$ angle D (by 1. 20.) therefore the two triangles ABC and AED are equiangular, and $AD : AE :: AC : AB$ (by 6. 4.) whence $AD : AC :: AE : AB$ (by 5. 16.) $Q.E.D.$
The utility of this problem is manifest: another it will be found in many parts of this



PROPOSITION X.

Problem.

To cut a given line (CD) into two parts (Cd and dD) which shall have the same proportion to each other as two given lines (EF and GH).

METHOD I.

Through the extremities of CD draw any two lines AC and BD parallel to each other, and make them respectively equal to the two given lines EF and GH (thus make BD equal to GH, and CA=EF). Lastly, draw BA cutting CD in d: then is CD so divided that the part Dd is to the part Cd as BD (equal to GH) is to CA (=EF).

For because CA is parallel to BD (by construction.) the angle A=angle B and the angle C=angle D (by 1. 29.) also the angle CdA=angle B dD (by 1. 15.) whence the triangles dCA and dBD are equiangular; wherefore Dd : DB :: Cd : CA (by 6. 4.) consequently Dd : Cd :: DB : CA (by 5. 16.)

METHOD II.

From one end of DC draw a line (CB) at pleasure, making any angle therewith, and on this line take CA=GH (one of the given lines) and AB=EF (the other).

Draw

Draw D B, and parallel to it draw A d cutting C in d, then the part D d is to the part d C as B A (=E F) is to C A (=G H.)

This is evident by 6. 2.

M E T H O D III.

Through each end of C D draw lines (C n and D m) parallel to each other (as in the first method) and take C n equal to any part of E F (*by 6. 9.*) as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. also take D m equal to a like part of G H.

Draw m n cutting D C in d; then the part D d is to the part C d as G H is to E F.

For by the argument used in the first method $Dd : dC :: Dm : Cn$; but $Dm : Cn :: GH : EF$ (*by construct.*) whence $Dd : dC :: GH : EF$ (*by 5. 11.*)

M E T H O D IV.

Draw from one end of C D the line C n (as in the second method) and on this line lay C m equal to any part of G H (*by 6. 9.*) as $\frac{1}{2}$, $\frac{1}{3}$, &c. and m n equal to a like part of E F.

Lastly, draw D n and d m parallel to it, cutting D C in d, and dividing it into two parts (D d and d C) which have the same proportion to each other, as the two given lines E F and G H.

For $nm : Cm :: Dd : dC$ (*by 6. 2.*)

Also $nm : Cm :: EF : GH$ (*by construct.*)

Whence $Dd : dC :: EF : GH$ (*5. 11.*)

The

The reason of performing this problem so many different ways, is because the whole practice of perspective is derived from one or other of these methods, as will be exemplified in pages 31, 32. 38. 95. 111. 123. 124. 126. 197. &c.

There is some variation from Euclid in the stating of this proposition, in order to render it more subservient to the business of perspective.

PROPOSITION XVIII.

Problem.

Upon a given line (AB) to describe a figure (ABCDE) similar to a given right lined figure (PQRST).

Draw PR and PS. Make (by 1. 31) the angles $ABC = PQR$; $CAB = RPQ$; $CAD = RPS$; $ACD = PRS$; $ADE = PST$ and $DAE = SPT$, and the thing is done.

DEMONSTRATION.

For it is evident from this construction, that the angles of the figure ABCDE are equal to the angles of the figure PQRST. Also since the said figures are composed of similar triangles, we have

$AB : BC :: PQ : QR$ (by 6. 4.) also

$PR : AC :: QR : BC$, and

$RR : AC :: RS : CD$, whence

$QR : BC :: RS : CD$ (by 5. 11.)

consequently $QR : RS :: BC : CD$; and in the

the same manner it may be proved, the other sides, which are about the equal angles, are proportional, wherefore the figures are similar by def. 2.

This problem is of very extensive utility: it will be referred to in page 213. 214. and 217.

Euclid's

The inclination of a line (A B) to a plane (F G) is the acute angle (A B C) formed

Euclid's Elements.

B O O K X I.

D E F I N I T I O N S.

III.

A Right line is perpendicular to a plane, when it is perpendicular to every line that can be drawn in that plane to the point on which it insists.

Thus A B is perpendicular to the plane G H, if it is perpendicular to all the lines that can be drawn in that plane to the point B.

IV.

One plane (B D) is perpendicular to another (F G) when any right line (I K) drawn in one of the planes (F G) perpendicular to their common section (D C) is perpendicular to the other plane B D.

V.

The inclination of a line (A B) to a plane (F G) is the acute angle (A B C) formed by
by

by the said line AB and a line (BC) drawn through the point (B) where the inclined line meets the plane and the point (C) where the said plane is cut by any perpendicular (AC) to that plane (FG) drawn from any point (A) in the inclined line.

VI.

The inclination of one plane (AF) to another (AD) is the acute angle (GHI) formed by two lines $(GH$ and $HI)$ drawn from any point (H) in their common section (AC) perpendicular to it, one upon one plane and the other on the other.

VIII.

Parallel planes are such as will never meet each other if produced every way.

PROPOSITION I.

Theorem.

One part of a right line cannot be in a plane and the other part out of it.

That is if one part, GE , of any right line GH , is situated in a plane (AD) the whole must be situated in that plane.

This follows from the nature of a plane, defined in the seventh definition of the first book.

This proposition is quoted in page 52.

PROPOSITION II.

Theorem.

If two right lines (AB and CD) cut each other, a plane (FG) may be extended by those lines, and all the parts of a triangle (DBC) are in the same plane.

DEMONSTRATION.

The first part of this proposition is rather an *axiom* or *postulate*, whose truth and practicability is self apparent. The latter is a conclusion immediately resulting from the former. For if we suppose a plane (FG) to be extended by the three angular points of any triangle (c D B) that plane must coincide with the sides, which are right lines (by *def. 7. book 1.*)

PROPOSITION III.

Theorem.

If two planes (AB and CD) cut each other, their common section is a right line.

DEMONSTRATION.

Let E and F be any two points in the common section of the planes A B and C D, and let a right line be drawn through these points.

Then

Then because the points E and F are situated in the plane AB, the right line EF will coincide with the plane AB (by def. 7. book I.) also because the said points E and F are situated in the plane CD. The right line EF will also coincide with the plane CD.

Wherefore the right line EF being wholly situated in both the planes AB and CD, it must be their common section.

This proposition is of singular use in almost all branches of the mathematics: in perspective it demonstrates that the perspective representation of a right line upon a plane, is a right line. Vide page 9th of the following work.

PROPOSITION IV.

Theorem.

If a right line (EF) is perpendicular to two other lines (AB and CD) in the point (E) of their intersection; it shall be perpendicular to the plane (AB) in which those lines are situated.

DEMONSTRATION.

1st, Take, from the intersection, (E) EA, EB, EC, and ED, all equal to each other.
2^d, Draw DA and BC, also through E draw any line, GH, in the given plane cutting AD and BC in the points G and H, and draw lines from F to the points A, B, C, D, G and H.

3^d, Now

3d, Now in the triangles AED and CEB , EA and ED are equal to EB and EC . Also the angle $AED = BEC$, whence (by 1. 4.) the angle $DAE = EBC$, and the side $DA = BC$.

4th, Again in the triangle GEA and BEH , the angle $GEA = BEH$ (by 1. 13.) Also the angle $GAE = EBH$ (by third step) and the side $EA = EB$ (by construction.) Whence $GA = BH$ and $GE = EH$ (by 1. 26.)

5th, Again, the four right angled triangles, EAF , EDF , EBF and ECF , having equal bases, and the perpendicular EF common, must have their hypotenuses, FA , FD , FB and FC , all equal (by 1. 4.)

6th, And since it was shewn that $AD = BC$ (in third step) it follows that the sides of the triangle DFA are respectively equal to the sides of the triangle BFC . Consequently angle $DAF = BCF$ (by 1. 8.) $= CBF$ (by 1. 5.)

7th, It has now been proved that $BH = GA$, (in fourth step) also that $BF = AF$, (fifth step) and that the angle $HBF = GAF$ (sixth step): wherefore the triangles GAF and HBF are equal in all respects (by 1. 4.) and the side $FG = FH$. Lastly, in the triangles GEF and HEF , it has been proved (fourth step) that $GE = EH$, and $FG = FH$, and FE is common to both. Wherefore the angle $GEF = HEF$. Consequently EF is perpendicular to GH (by def. 10.)

In the same manner it may be shewn that EF is perpendicular to every line that can be

h
drawn

drawn on the plane $A B C D$ through the point E ; and therefore is perpendicular to that plane (by 11. def. 3.) Q. E. D.

This proposition is quoted in pages 172 and 260.

PROPOSITION V.

Theorem.

If three lines (BC , BD and BE) meet in a point (B) and another line (AB) passing through the same point (B) is perpendicular to them all, these three lines (BC , BD and BE) are in the same plane (GH).

DEMONSTRATION.

1st, For if this is denied; suppose BC and BD are in any plane, GH , and that BE falls below the said plane.

2^d, Let BI be a plane passing by BA and BE , and cutting the plane GH in the line BF .

3^d, Because BA is perpendicular to BC and BD , it is also perpendicular to the plane GH (by 11. 4.) and consequently to BF (by def. 3. book 11.) but BA is perpendicular to BE (by supposit.) wherefore the angle EBA and FBA , being both right angles, are equal, which is impossible.

Consequently BE is not below the plane GH ; and the same reasoning will prove it is not above GH ; wherefore it is in the plane GH . Q. E. D.

This proposition will be wanted to prove the next.

P R O-

PROPOSITION VI.

Theorem.

If two lines (A B and C D) are perpendicular to the same plane (E F) they are parallel.

DEMONSTRATION.

1st, Draw the line D B on the plane E F, joining the A B and C D. Then since each of these lines are perpendicular to the plane E F (by supposition) they will be perpendicular to D B (by def. 3. book 11.) consequently the angle CDB + ABD = two right angles; wherefore, if we can also prove, that the lines A B and C D are situated in the same plane, it will follow that they are parallel (by 1. 28.)

2^d, On the plane E F draw D G perpendicular to D B (by 1. 11.) and equal to A B, and draw G B, G A and D A.

3^d, In the triangles B D A and B D G, the sides B D and A B, and the included (right) angle A B D, are respectively equal to B D and D G, and the included (right) angle B D G, whence (by 1. 4.) the side B G = D A.

4th, The triangles G A B and G A D, have the three sides of the one respectively equal to the three sides of the other: for A B = D G (by the second step) G A is common, and B G is equal to D A (by the third step) wherefore the angle A D G = angle A B G, which is a right angle (by supposit. and def. 3.)

5th. The line DG is perpendicular to DB (by second step) and to DA (by the fourth) and to DC (by suppos. and def. 3.) wherefore those lines are all in the same plane (by 11. 5.) But the lines DA and DB are in the same plane with AB (by 11. 2.) consequently AB and CD are in the same plane, and so are parallel (by first step). Q. E. D.

This proposition serves to demonstrate others, and will be quoted in page 261.

PROPOSITION VII.

Theorem.

All right lines (ab , cd , &c.) that can be drawn to cut two parallel lines (AB , CD) are in the same plane with those parallels.

This is evident from Def. 7. book I, for the right line ab , must coincide with the plane in which the points a and b are situated.

This proposition will be wanted in page 251.

PROPOSITION VIII.

Theorem.

If one (AB) of two parallel lines (AB , CD) is perpendicular to a plane (EF) the other will be so likewise.

DEMONSTRATION.

1st, Draw DB in the plane EF , joining the two parallel lines AB and CD . Then because AB is perpendicular to DB (by def. 3.) CD must also be perpendicular to DB (by 1. 29.)

2d,

2d, On the plane EF , draw DC perpendicular to DB (by 1. 11.) and equal to AB , and draw DA , GA and GB .

3d, Then the two triangles DBA and DBG , having two sides DB and BA with the included (right) angle DBA in one; equal respectively to the sides DB and DG , with the included (right angle) BDG in the other, GB must be equal to DA (by 1. 4.)

4th, Again the triangles AGD and ABG have the sides of the one respectively equal to the sides of the other. For $GB=AD$ (by the third step) AG is common, and $DG=AB$ (by the second step) wherefore the angle ADG is equal to the angle ABG (by 1. 8.) which is a right angle (by supposit. and def. 3.) wherefore DG being perpendicular to the two lines DB and DA , is perpendicular to the plane of those lines (by 11. 4.) and consequently to the line DC (vide 11. 7.) wherefore CD being perpendicular to DB (by the first step) and to DG (by the fourth step) is perpendicular to the plane of those lines, EF (by 11. 4.) Q. E. D.

This theorem serves to demonstrate the 9th, 14th and 18th of this book; and will be wanted in page 175.

PROPOSITION IX.

Theorem.

Lines (AB and CD) which are parallel to the same right line, (EF) and not in the same plane with it, are parallel to each other.

DEMON-

DEMONSTRATION.

Through any point G , in the line EF , draw GH , at right angles to EF , in the plane $(CDEF)$ passing by EF and CD (by prop. 11. book 1.) Also draw GI , at right angles to EF , in the plane $(ABEF)$ passing by EF and AB . Then because EF is perpendicular to GH and GI , the said line EF will be perpendicular to the plane HI , passing by those lines (by prop. 4. of this) and because AB is parallel to EF , it also will be perpendicular to the plane HI , (by prop. 8. of this); and for the like reason CD is perpendicular to the same plane HI , consequently AB and CD are parallel (by prop. 6. of this). Q. E. D.

This proposition will be wanted in the demonstration of the next, the 15th, and in pages 42. 231. and 233.

PROPOSITION X.

Theorem.

If two lines (AB and BC) meeting in a point (B) are respectively parallel to two other lines (DE and EF) meeting in a point (E) and not in the same plane with the two first, the angles (B and E) formed by these lines will be equal.

DEMONSTRATION.

Take $AB=DE$ and $BC=EF$ and draw AC and DF ; also draw AD , BE and CF .

Now

Now because AB is equal and parallel to DE (*by supposit.*) AD is equal and parallel to BE (*by 1. 33.*) For the same reason CF is equal and parallel to BE , consequently AD is equal and parallel to CF (*by 11. 9.*) wherefore AC is equal and parallel to DF (*by 1. 33.*) And the sides of the triangle ABC are respectively equal to the sides of the triangle DEF , consequently the corresponding angles B and E are equal (*by 1. 8.*) $Q. E. D.$

This proposition will be quoted in pages 12. 43. 54. and 171.

PROPOSITION XIV.

Planes (EF and GH) to which one and the same line (AB) is perpendicular, are parallel to each other.

DEMONSTRATION.

From any point (C) in the plane EF , let CD be drawn parallel to AB , and draw AC and BD .

Then because CD is parallel to AB , it is perpendicular to both the planes EF and GH (*by 11. 8.*) wherefore the figure $ABCD$ is a rectangular parallelogram (because the angles A, B, C, D are right); whereof the opposite sides BA and CD are equal (*by 1. 34.*)

By

By the same argument all other perpendiculars, terminated by the two planes, are equal, to AB ; consequently these planes can never meet and must be parallel (by 11. def. 8.)

This proposition will be quoted in pages 79, 175, and 177.

PROPOSITION XV.

Theorem.

If two lines (AB and AC) meeting each other, be respectively parallel to two other lines (DE and EF) also meeting each other, and not being in the same plane with them, the planes (MN , and KL) extended by these lines will be parallel.

DEMONSTRATION.

Let AG be perpendicular to the plane MN , meeting the plane KL in G , in which last plane let GH and GI be drawn parallel to DE and DF .

Now because GI is parallel to DF , and AC is also parallel to DF (by supposit.) AC is parallel to GI (by 11. 9.)

For the same reason AB is parallel to GH . But AG is perpendicular to AC (by const. and def. 3.) wherefore it is also perpendicular to GI (by 1. 29.) and for a like reason AG is also perpendicular to GH , and consequently to the plane KL (by 11. 4.)

Seeing then that AG is perpendicular to both the planes MN and KL , those planes are parallel (by 11. 14.)

This

This proposition will be referred to in pages 41. 55. 228. 230. 231. and 250.

PROPOSITION XVI.

Theorem.

If a plane (EFGH) cuts two parallel planes (AB and CD) their common sections (EF and GH) will be parallel.

DEMONSTRATION.

For because the line EF is situated in the plane AB, it will agree with that plane if produced to infinity (by def. 7. 1.) In like manner the line HG will ever agree with the plane CD. Wherefore seeing these planes are parallel (or can never meet each other); the lines EF and GH can never meet each other; but these lines EF and GH are also both situated in the plane EFGH (by supposition) consequently, since they are in the same plane, and can never meet, they are parallel by def. 35. I.

This theorem is very often wanted in demonstrating the principles of perspective, as will be seen in pages 42. 179. 180. 181. 228. 229. and 230.

PROPOSITION XVII.

Theorem.

If a line (PQ) is perpendicular to a plane (AB) any plane (ED) passing by that line, will be perpendicular to the same plane (AB).

i DEMON-

D E M O N S T R A T I O N.

Draw any line IK in the plane ED perpendicular to the common section (CD) of the two planes AB and ED . Then the angle DIK being a right angle, is equal to DQP , wherefore KI is parallel to QP (*by* 1. 28.)

But PQ is perpendicular to the plane AB (*by supposit.*) whence KI is also perpendicular to the same plane (*by* 11. 8.) By the same reason all lines drawn in the plane ED perpendicular to the common section (CD) is perpendicular to the plane AB , wherefore those planes are perpendicular to each other (*by* 11. def. 4.)

C O R O L L A R Y.

Hence a line, standing at right angles to one of two perpendicular planes, at any point (I) in their common section (CD) must be in the other plane.

For the line IK , in the plane AD , is perpendicular to the plane AB ; besides which, another line perpendicular to AB , from the same point I , cannot be drawn, for if that were possible they must be both parallel to PQ (*by* 11. 6.) which is impossible (*by* ax. 12. book 1.)

This corollary is of use to demonstrate the next proposition. The proposition itself is extremely useful in perspective, as will appear in pages 172. 175. 177. 182. 259. and 267.

P R O-

PROPOSITION XIX.

Theorem.

If two planes (AB, CD) cutting each other, be both perpendicular to a third plane (GH) their common section, will also be perpendicular to the said plane (GH.)

DEMONSTRATION.

For from the extremity (F) of the common section, let the line FE be erected perpendicular to the plane (GH), which line will coincide with both the planes AB and CD (by cor. to 11. 18.) and must consequently be their common section, wherefore their common section is perpendicular to GH. Q.E.D.

This proposition will be wanted in pages 172. 177. 179. 184. and 260. of the ensuing work.

Some Errors of the Press and the Engraver are almost unavoidable in works of this kind: if the Reader corrects with his pen the following, it is presumed there are no others capable of giving him the least embarrassment.

N. B. b signifies *from the bottom*, and t signifies *from the top*.

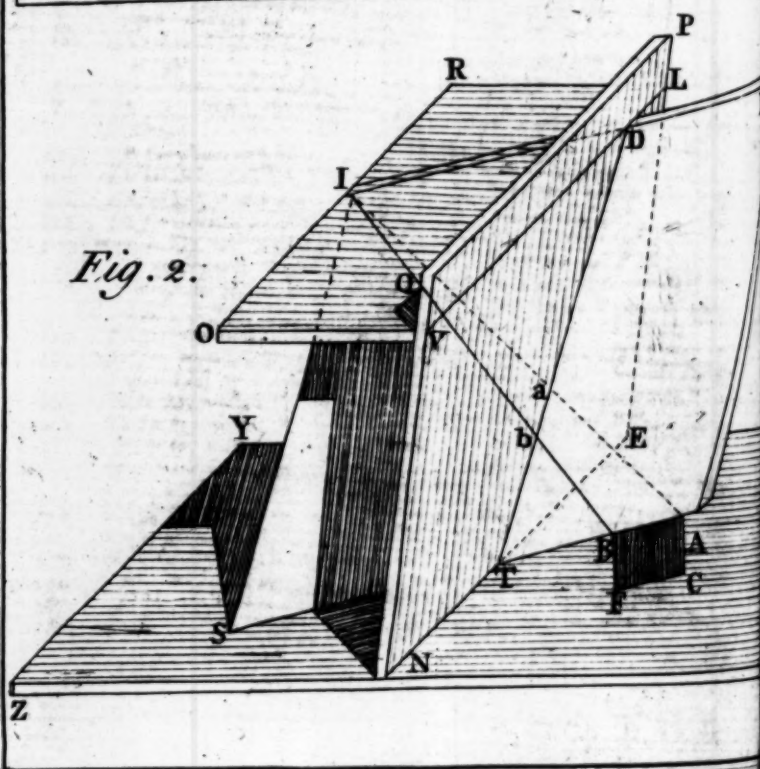
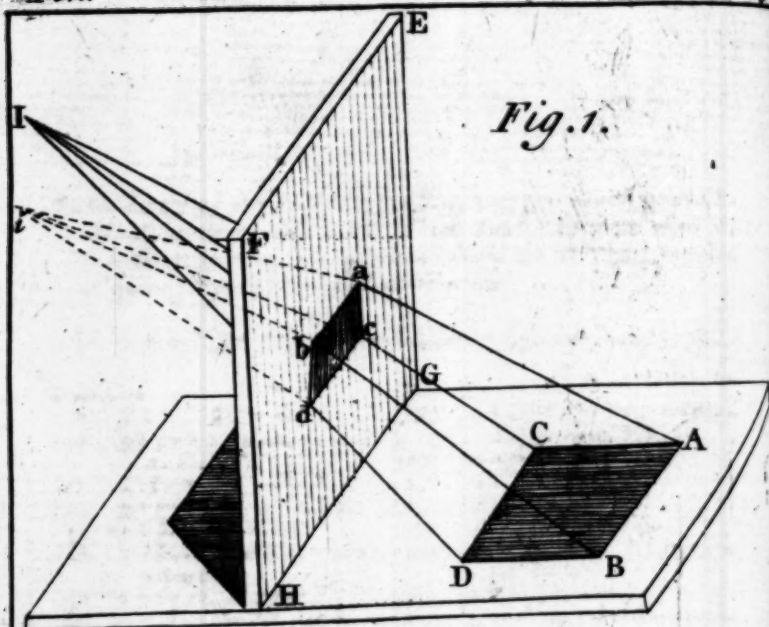
Page	Line	Page	Line
4	8 b for ABCD read ABCF.	187	3 t and 6 t for left, read right, and for right, read left.
8	9 b for E. 1. 31 note, read E. 1. ax. 12.	190	7 b for 62, read 63.
21	8 t for ATN, read AKE.	191	2 t for 63, read 64.
21	9 t for aTN, read aKE.	191	8 t for 62, read 63.
32	11 b for nd, read nV.	192	4 t for LH, read H.
48	1 b for original line, read: entering line.	200	3 t after (1 fig. 8.) insert, is laid down.
57	6 b for c b, read c a.	201	13 b for J a, read J d.
66	13 b after through, insert, the center of.	201	2 b cross out, as in this example.
66	5 b for more or less, read less or more.	205	4 b for plan; read plano.
68	6 b for cure, read curve.	208	13 t for 66, read 65.
80	7 b for distance, read disposition.	212	13 b for 54, read 64.
106	5 b for 3 b, read 3 d.	212	8 b for taught, read sought.
106	2 b for A b, read A d.	215	12 b for v, read u.
108	3 b for left, read right.	219	8 b and 9 b for z, read Z.
108	2 b for right, read left.	233	12 and 13 t for SV, read 6V.
109	10 t for O f, read O g: and cross out the words "equal to the thickness of the wall, and f g."	223	6 b for E 3, 3, read E 3. 1.
119	9 b for vanishing, read visual.	235	10 t for B C d, read B C b.
123	10 t for appearance, read representation.	245	11 t for 72, read 72*.
126	18 t for 44, read 46.	249	5 t for 73, read 73*.
128	6 b for first, read last.	251	3 t for PN, read P M.
131	15 t for 50, read 51.	264	12 t cross out these words, "in this side b c I suppose there is a door into the street."
135	3 t and 5 t for b, read c.	264	16 b after a b, insert "and in this side c d I suppose there is a door into the street."
135	13 t for 13, read 131.	268	5 t for A p, read A P.
138	9 b for P, read p.	269	9 t for A p, read A P.
163	18 b for g l h, read g l h.	270	1 t for i g, read i g.
177	15 b and 20 b for A I, read A i.	279	1 t for shorter, read "not shorter."
181	1 b for its, read the.	280	7 b and 9 b for N W b W, read N W b N.
182	1 t after are, insert, parallel to.	280	7 b for I K, read I k.
186	14 t for 50, read 61.	283	17 t for grows, read grow.
		293	3 b for ABCD, read ABED.

THE

Some of the Press and the Engineer are almost un-
able to work of this kind if the Reader connects with
pen the following, it is presumed there are no other
of giving him the least embarrassment.

N. B. Signifies from the bottom, and L. Signifies from the top.

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187	3	187	3
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ELEMENTS

O F

Linear Perspective.

PRELIMINARY OBSERVATIONS.

IF the rays of light which flow from a luminous body fall on any object, they are again reflected from that object, every way, in right-lined directions; and if any of these reflected rays encounter the eye of a spectator, they are, by the humours of the eye made to converge on the retina, * and excite a sensation which is called the *appearance* of that object.

If the rays proceeding from any enlightened object to the eye are intercepted by a transparent plane, the section of these rays with that plane will form thereon, what is
B called

* The nature of vision, and the construction of the eye will be explained hereafter.

called the *perspective representation* of that object.

Thus, suppose E F G H (fig. 1.) is a transparent plane, through which a spectator looks at the figure A B C D, the rays coming from every part of which to the eye, at I, form, by their intersection, the figure a b c d, which is the *perspective representation* of the *original object* A B C D on the transparent plane, which is called *the plane of the picture*.

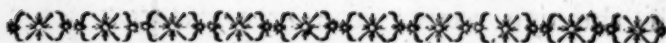
From this account it is evident, that when a person looks through a glass-window at the objects out of doors, he sees the *perspective representation* of those objects delineated on the glass; which in this case is the *plane of the picture*: and if this representation was traced over, and the outline filled up with the colours and shades of the original objects; the pane of glass would at all times, and in all places, excite the same ideas in the mind which the natural objects themselves did when they were first viewed; provided the eye is always placed in the same *position* and at the same *distance with respect to the glass*: for the rays will then come from this *picture* to the eye, exactly in the same direction as they did from the *original objects*.

Thus the eye, when placed at I (fig. 1.) views the *representation* a b c d by the rays, a I, b I, c I, d I, &c. which enter it exactly in the same direction as the rays A I, B I, C I, D I, &c. which come from the original object.

It

It is very requisite to be remembered, that if the eye views the *representation* $a b c d$ from any other point than I , as suppose at i , then the rays ai , bi , ci , di , &c. which come from the several parts of the *representation*, will not be the same as those which come from the original object $A B C D$, and therefore the *representation* $a b c d$ will not convey the same idea as the *original figure*, $A B C D$ itself does, when viewed from I .

Since the *perspective representation* of any object is the *intersection of the rays* which are reflected from that object to the eye *with the plane of the picture*; If the position and distance of the eye I (fig. 1.) and of the *original object* $A B C D$, with respect to the *picture plane*, $E F G H$, be given or assumed; and if we can by any method of reasoning, find the points, a , b , c , d , where the rays coming from this *original figure* to the eye will cut the picture plane; we may then delineate the *perspective representation* of any object whose figure and position exists only in the imagination. The rules which geometrical reasoning discovers for this purpose, constitute what is called THE ART OF PERSPECTIVE.



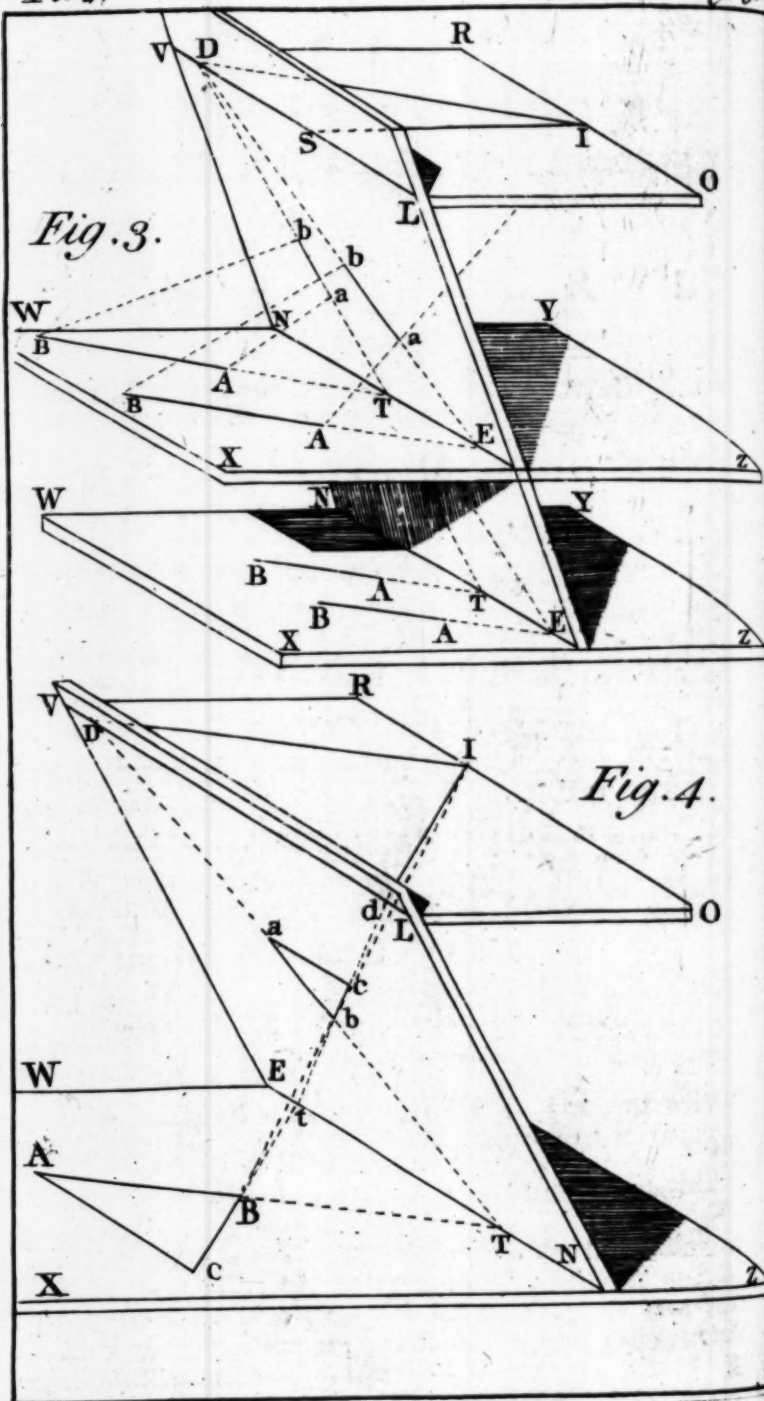
C H A P T E R I.

Definitions and principles for delineating the representation of right-lined figures, situated in planes which are not parallel to the plane of the picture.

I. **L**ET PQEN (fig. 2.) represent a transparent plane, through which an eye placed at I looks at any right-lined figure, ABCF, situated in any indefinite plane WXYZ: this transparent plane is called the *picture plane*: the figure ABCD is called the *original object*; and the indefinite plane WXYZ in which the original object is situated, is called the *original plane*.

II. If any original plane (WXYZ) is not parallel to the picture, it will, if produced, cut it in a right line (by E. 11. 3. *) as in EN, that line then, which is the intersection

* N. B. E. always signifies Euclid, the first Number shews the Book, and the latter the Proposition: as Euclid, Book II, Prop. 3.-----No Proposition is ever referred to but what is contained in the geometrical introduction prefixed to this work.



tion of the original plane and the picture, is called *the Entering Line* of that original plane.

III. Suppose a plane (LVOR fig. 2.) to pass through the eye (I) parallel to the original plane (WXYZ) this plane is called *the Vanishing Plane of the original plane*; and the line (VL) in which it cuts the picture, is called *the vanishing line of the original plane*.

T H E O R E M I.

IV. All *original planes* (WXYZ, WXYZ fig. 3.) which are parallel to each other, have the same *vanishing line*, (VL).

D E M O N S T R A T I O N.

For the plane LVOR, passing through the eye (I) parallel to any one, is parallel to all these *original planes*, and therefore the line VL where it cuts the plane of the picture, is their *vanishing line*, (by § 3.)

C O R O L L A R Y.

V. Any original plane which is parallel to the picture can have neither *entering* nor *vanishing line*; because neither it, nor a plane parallel to it and passing through the eye can cut the picture.

T H E O R E M II.

VI. The *entering and vanishing lines* of any *original plane* are always parallel to each other.

D E M O N S T R A T I O N .

For the entering and vanishing line EN and VL (fig. 3.) are formed by the intersection of the plane of the picture ENVL, with the original plane, WXYZ, and its parallel LVOR (by § 2 and 3.) but when a plane cuts two parallel planes the sections are parallel to each other (by E. 11. 16.) Q. E. D.

VII. Draw a line, IS (fig. 3.) through the eye I, along the vanishing plane, LVOR, perpendicular to the vanishing line LV (by E. 1. 12.) and the point S, where this line cuts the picture ENVL is the *center of the vanishing line*; and the perpendicular line IS is called the *distance of the vanishing line*.

VIII. Let us now suppose the eye I (fig. 2.) to be contemplating any line AB; which if not parallel to the picture, will, if produced, cut it somewhere, as in T. Any line thus under contemplation is called an *original line*; and the point, T, where it cuts the picture, is called its *entering point*.

IX. If a line, ID (fig. 2.) is drawn thro' the eye I, parallel to any *original line*, AB, and produced till it cuts the *picture* in D, it is called the *distance of that original line*, and the point D, where it cuts the picture, is the *vanishing point of that original line*, AB.

T H E O R E M III.

X. All *original lines* (AB, AB, AB, AB, &c, fig. 3.) which are parallel to each other, have

have the same *Distance* (DI) and *Vanishing point* (D).

DEMONSTRATION.

For the line DI, drawn through the eye, I, parallel to any one of the original lines, AB, AB, AB, AB, is parallel to them all; (by E. 11. 9.) wherefore the line DI is their *distance line*, and the point D where it cuts the picture, is their *vanishing point* (by § 9).

COROLLARY.

XI. Hence an original line which is parallel to the picture can have no *entering* nor *vanishing point*: for neither it, nor its *distance*, (which is parallel to it, § 9.) can ever cut the picture.

XII. The extremities, A, B, (fig. 2.) of any original line, AB, together with the point I, where the eye is placed, form three points; but a plane, ASIH may be extended by any three points, A, B, I (by E. 11. 2.) consequently a plane may be extended by any original line, AB, and the eye I; which plane, ASIH, is called the *visual plane of that Original Line*.

THEOREM IV.

XIII. The *Distance*, ID (fig. 2.) of any *Original Line*, AB, is the intersection of its *visual plane*, ASIH, with the *vanishing plane*, LVOR, of that *original plane* (WX
B 4 Y Z)

YZ) in which the said *original line*, AB, is situated.

DEMONSTRATION.

For the visual plane, ASIH must cut the original plane, WXYZ and its parallel LVOR, by two parallel lines, AT, ID (by E. 11. 16.) but it cuts the original plane, WXYZ, along the original line, AB (by § 12.) and since the visual plane passes through the eye (by § 12.) it must therefore cut the vanishing plane, LVOR, in a line, ID, passing through the eye I, parallel to the original line, AB; but ID, *the distance of the original line*, is a line drawn through the eye I, parallel to the original line, AB (by § 9.) therefore the distance, ID, is the section of the visual plane, ASIH, with *the vanishing plane*, LVOR. For if it were not, then through the same point I, two lines might be drawn parallel to the same right line, AB, which is impossible (E. 1. 31. note).

XIV. The right line (TD, fig. 2.) joining the entering point (T) and the vanishing point (D) of any original line (AB) is called *the visual line* of that original line.

T H E O R E M V.

XV. *The visual line*, TD (fig. 2.) of any original line, AB, is the section of the *visual plane*, AHIS, and the *picture*, ENPQ.

DEMON-

DEMONSTRATION.

For the entering point, T, is in the picture and visual plane (by § 8. and 12.) also the vanishing point, D, is in the picture (by § 9.) and the visual plane (by § 13.) wherefore the line TD, joining these points, TD, must lie in both the planes, ENPQ and AHIS, and consequently must be their intersection, (E. 11. 3) Q.E.D.

T H E O R E M VI.

XVI. The *perspective representation*, *a b* (fig. 2.) of any *original line*, *A B*, is a part of its *visual line*, *T D*.

DEMONSTRATION.

For the perspective representation of any original line, *A B*, is the section of the rays *A I*, *B I*, &c. which come from every part of that line to the eye, *I*, with the picture: (*Vide Prelim. Observat.*) But all these rays that can be drawn from the original line, *A B*, to the eye, must lie in the visual plane, *A I S H* (by Art. 12.) therefore they must cut the picture in the same line (*T D*) in which it is cut by the visual plane; and consequently must cut off a part, *a b*, of the visual line, Q. E. D.

C O R O L L A R Y.

XVII. From thence it is plain, that the *representation* (*a b*, *a b*, *a b*, *a b*, fig. 3.) of all

all *original lines* (A B, A B, A B, A B) which are parallel to each other, tend to the same point, D, which is their *vanishing point*. For since the *Representation* of every original line is a part of its visual line (by § 16.) and since the visual line of any original line (A B) is a right line joining the entering and vanishing points, T and D of that line; (by § 14.) it follows, that the perspective *Representation* of any original line tends to its vanishing point; but all original lines, A B, A B, A B, A B, which are parallel to each other, have the same vanishing point, D (by § 10.) therefore their perspective representation, a b, a b, a b, a b, must tend to their *vanishing point* D.

T H E O R E M VII.

XVIII. Let W X Y Z (fig. 4.) be any original plane, on which there is drawn the lines A B, C B. Let these lines be produced till they cut the picture in the points, T and t, which are there *entering points* (by § 8.) Let the eye be at I through which the plane, V L O R passes parallel to the *original plane*, W X Y Z, then is the line (V L) its *vanishing line* (by § 3.) through the eye, (I). Let there be drawn the *distance lines*, I D, and I d (§ 9) parallel to their respective originals, A B and C B; then will D and d be their *vanishing points*. Join the points T, D and t, d, then will T D and t d be the *visual*
lines

lines of the originals, CB, BA (§ 14.) Then I say; The intersection *b* of any two visual lines, (TD and *t d*) is the perspective representation of The intersection B, of their original lines, (AB, BC.)

DEMONSTRATION.

The *perspective representation* of any point, B, is that point where the ray, BI, coming from it to the eye, cuts the picture (*Vide Prelim. Observ.*) But all the rays that can be drawn from the line AB, to the eye, I, must intersect the picture in its visual line, TD (by § 16.) consequently the point B, being situated in the line, AB, its representation must be somewhere in the *visual line* TD, by the same reasoning, since the point B, is also situated in the line BC; its representation must be somewhere in the line *t d*, which is the visual line of BC. Having therefore proved that the representation of the point B, must lie in both the lines TD and *t d*, it must necessarily lie in their intersection, *b*. QED.

T H E O R E M VIII.

XIX. The angle (ABC, fig. 4.) formed by any two *original lines* (AB, BC) is equal to the angle (DI*d*) formed at the eye (I,) by the *distance lines* (ID, Id) of those *original lines*.

DEMON-

DEMONSTRATION.

For the lines AB , BC and DI , Id , being respectively parallel to each other (by § 9.) the angles, ABC and $DIId$ must be equal to each other (by E. 11. 10.)

THEOREM IX.

XX. The angle (ATE) fig. 4. which any *original line*, (AB) produced till it meets the picture, makes with the *entering line*, EN , of the plane, $WXYZ$, in which it is situated; is equal to the angle (IDL) which the *distance* (DI) of that line, makes with the *vanishing line* (VL) of that *original plane*.

DEMONSTRATION.

The lines AT , TE are respectively parallel to the lines ID , DL (by § 9. and 6.) therefore the angles, ATE , IDL are equal (by E. 11. 10.)

COROLLARY.

XXI. Hence the *center* and *distance* of the *vanishing line*, is the *vanishing point* and *distance* of any *original line* that is perpendicular to the *entering line* (by § 7. and 20.)

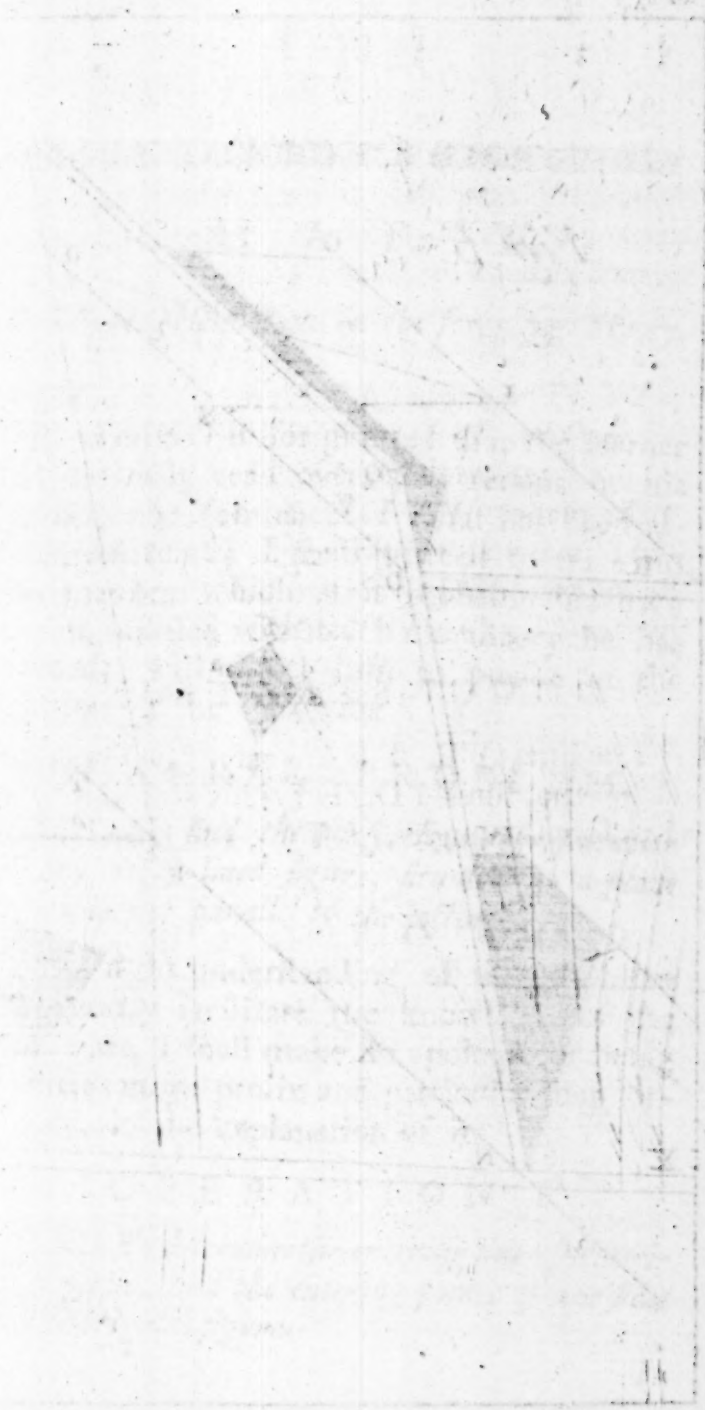
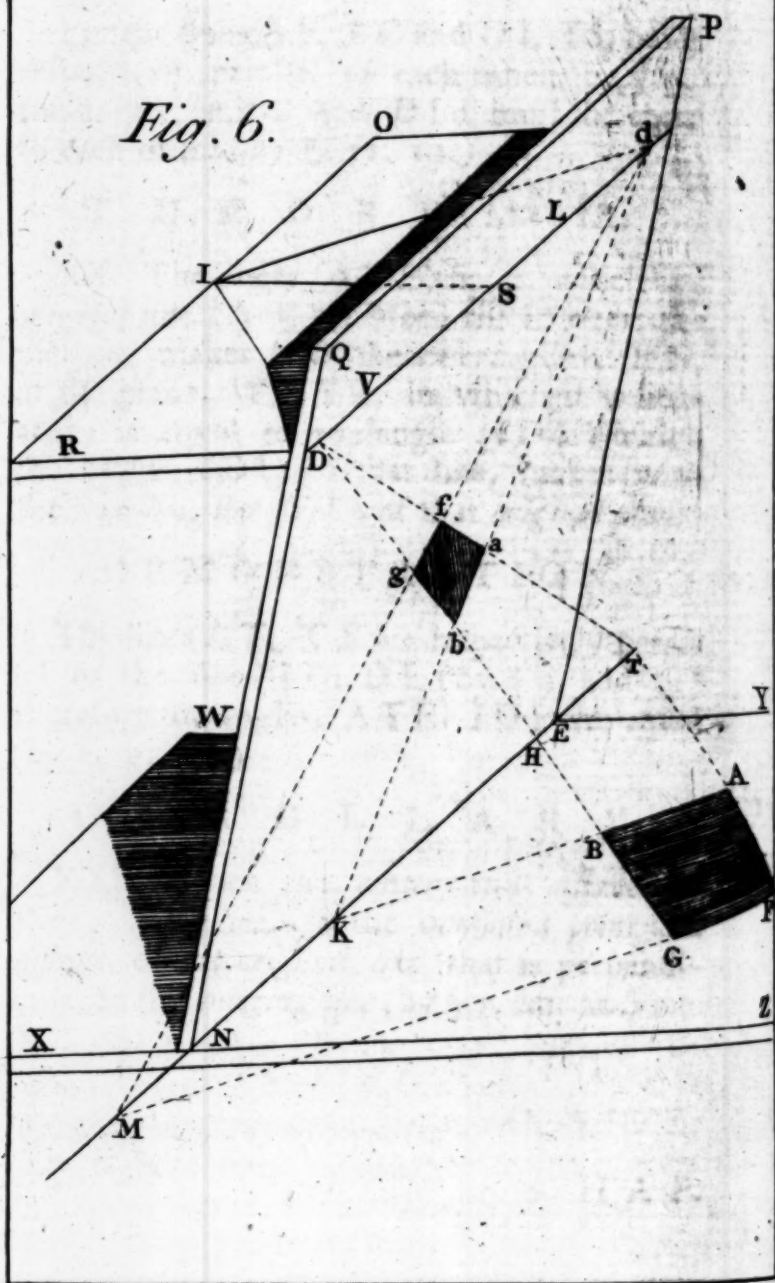


Fig. 6.



C H A P. II.

Practical Observations on the foregoing Theory.

TAKING it for granted that the learner has carefully read over, and retains in his memory the substance of what has been already delivered; I shall proceed to trace out the manner which it is probable his own reason, assisted with the little theory he has learned, will direct him to pursue in the solution of the following

G E N E R A L P R O B L E M.

XXI. *To find the perspective representation of any right-lined figure, situated in a plane which is not parallel to the picture.*

As a right understanding of this problem will greatly facilitate the knowledge of the whole art, I shall make no apology for being a little more prolix and particular than ordinary, in the explanation of it.

O P E R A T I O N I.

XXII. *To determine the entering line of the original plane, and the entering points of the sides of the original figure.*

In

In the first place we will imagine the figure which the learner would represent in perspective to be a parallelogram, as $ABFG$ (fig. 6.) whose shape and dimensions are drawn out on any plane surface, as a piece of paper or paste-board, represented in the scheme by the plane $WXYZ$. On this plane (which is the original plane by § 1.) another plane $EPQN$,¹ must be erected to represent the plane of the picture. This may be done in any position, with respect to the original plane, and the object $ABFG$, which is most agreeable to his fancy; the line EN , which we suppose is allotted to be the line in which the picture $ENQP$ shall cut the original plane $WXYZ$, is the entering line (by § 2). If this line is not parallel to either of the sides of the *original figure* AB , GB , AF , FG , these will, if produced, cut the entering line EN , in the points T , H , K , M , which are the entering points of the sides AF , BG , AB , FG , of the original figure (by § 8.)

OPERATION II.

XXIII. *To determine or assume the vanishing line, its center and distance, and of consequence the place of the eye.*

Before the learner can proceed any further, he must determine the position of the eye, I: to do this draw at pleasure, the line
VL

VL on the plane of the picture ENPQ parallel to (EN) the *entering line*: (by E 1. 31.) then this line VL, may be the vanishing line of the original plane. (by § 6.) Along this line a plane VLOR, must be passed parallel to the original plane WXYZ. And this plane will be the vanishing plane. (by § 3.) The learner may proceed to assume any point, S, in the vanishing line VL, for the center of the vanishing line (§ 7.) at this point he must draw the line SI along the vanishing plane VLOR, perpendicular to the vanishing line: (by E. 1. 11.) in this line he may at pleasure assume a point, I, for the place of the eye. (Vide § 7.)

OPERATION III.

XXIV. *To find the vanishing points of the sides of the original figure.*

The place of the eye, I; (fig. 6.) being thus determined, the next thing to be done is to find the vanishing points of the sides of the original figure ABFG. To do this, measure the angle ATN, which any of the original lines, as AF, makes with the entering line TN; then on the vanishing plane VLOR, draw through the point I, the line DI, making the angle DIS, with the distance IS of the vanishing line, equal to the complement of the angle ATN; then the point D, where
this

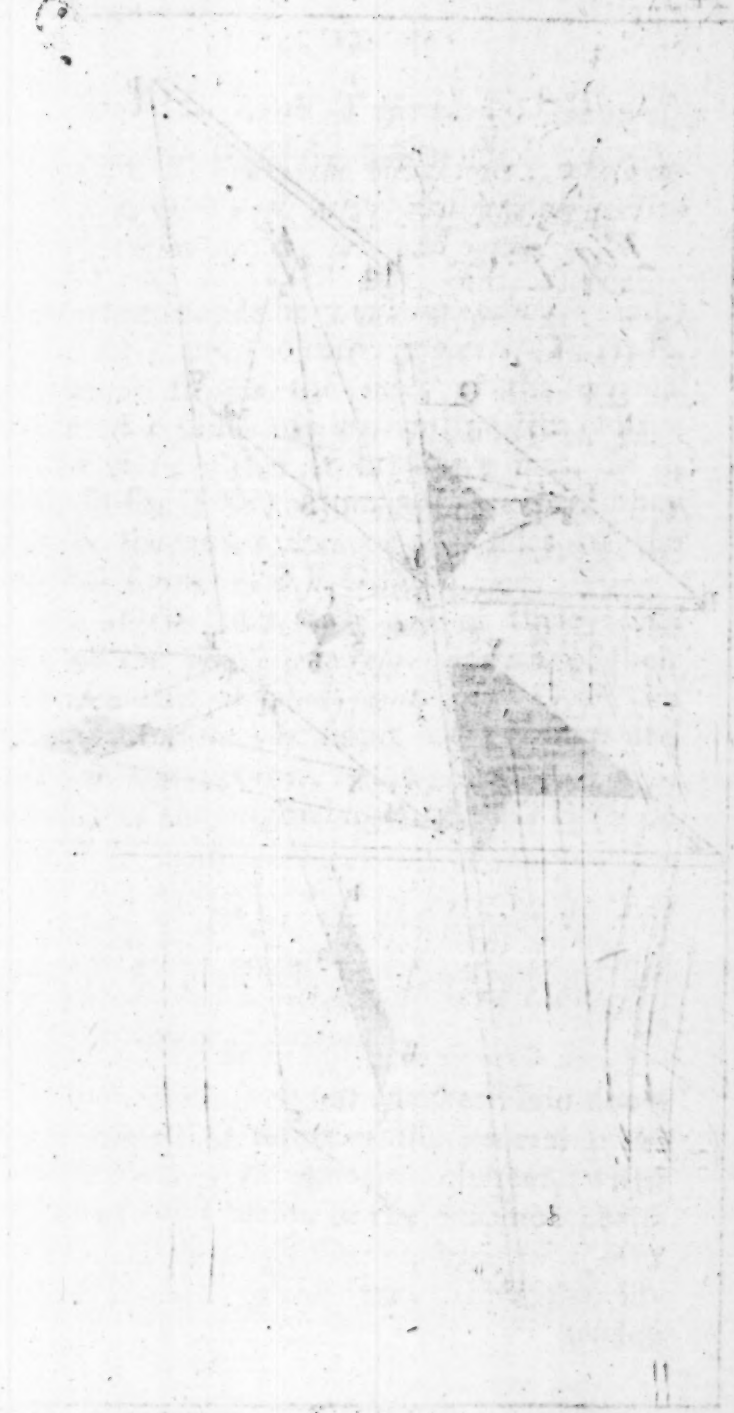
this line DI cuts the picture, is the vanishing point of the original line, AF .

DEMONSTRATION.

Because the triangle DSI is right-angled, the angle SDI is the complement of the angle DIS ; (by E. 1. 32. Cor. 3.) and the angle ATN is equal to the complement of DIS by construction; wherefore the angle ATN is equal to the angle SDI , (E 1. ax. 1.) and the line DI is the *distance line* of the original line, AF (by § 20.)

XXV. Having determined the *vanishing point* D , of any one side AF , of the *original figure*, the *vanishing point* d , of any other side, AB , may easily be found. For if a line, dI , be drawn through the eye I , on the *vanishing plane*, $VLOR$, making the angle Did , equal to the angle AFG or BAT (which the original lines, AF and AB make with each other) then the point d , where this line dI cuts the picture, is the *vanishing point* of the line, AB (by § 19.)

Since the original figure $ABFG$ (fig. 6.) is a parallelogram, the side GB , will have the same vanishing point D , with its parallel side AF , and the side FG has the same vanishing point as the side AB (by § 17.)



O P E R A T I O N IV.

XXVI. *To draw the visual lines ; whose intersections with each other give the perspective representation of the original figure.*

Having found the *vanishing points* (D, d,) (§ 24 25.) and the *entering points* (T, H, K, M, by § 22) of the sides of the *original figure* (A F G B) the learner has already been taught (§ 14.) that if he draws lines, M d, K d, H D, T D to join those points, they will be the *visual lines* of the sides of the original figure A B F G.

The intersections a, b, g, f of these *visual lines* are the *perspective representations* of their correspondent *original points* A, B, G, F (by § 18) wherefore the figure a b g f contained between those points, is the *perspective representation* of the original figure A B G F (Vide § 16.)

It will be proper for the learner to read these operations repeatedly over till he clearly comprehends the practice of each, and the reasons on which they are founded.

The preceding operations were laid down as the immediate result of the general principles delivered in the first chapter, when applied to the solution of the proposed problem (*to find the perspective representation of a given right-lined figure*) they are rather intended

tended to exemplify those general principles, than proposed as a refined mode for practice. They are only to be esteemed as the first attempts at a practice of perspective; and therefore like all other primitive endeavours are destitute of that elegance and ease which is the result of continued observation and experience.

We are in these operations obliged to make use of three different planes, which if we could find no means of working without, would render the act of perspective very operose and troublesome; but a few remarks on each of the operations will dictate to us, that one plane only will answer the purposes of those three.

REMARKS on the first OPERATION.

XXVII. It was first supposed that the *original figure* was drawn in its proper proportion on the *original plane* and that the position of the entering line was given or assumed. (*Vide* § 22.) Suppose W X Y Z (fig. 7) to be the *original plane*, on which is drawn the *original figure* A B F G; the position of the *picture* p q r s, is next determined, and the *entering points* T, H, K, M are found by producing the sides of the *original figure* till they cut the *entering line* E N, which is the first operation: the *entering points*, therefore, depend on the situation of the *original figure* (A B F G) with respect to the *entering line*
E N)

(E N) but the *entering line* E N, is situated in the plane of the *picture* p q r f, as well as in the *original plane* W X Y Z (by § 2.) If then another *original figure* a b g f is drawn on the *picture plane* p q r f, exactly of the same shape and dimension, and in the same situation with respect to the *entering line* E N, as that, ABGF, which was drawn on the *original plane* W X Y Z; then the sides a b, b g, g f, f a, of this figure being produced, will cut the *entering line* E N, in the same points K, H, M, T, as the sides of the figure A B G F (drawn on the *original plane*) and therefore will give the same *entering points*.

D E M O N S T R A T I O N.

According to the supposition the angle A T N, (drawn on the *original plane*) is equal to the angle a T N: (drawn on the *picture*) and the line A T is equal to T a, also because the figures are equal and similar, A F = a f, f g = F G, g b = G B, and b a = B A, and the corresponding angles of the two figures are also equal; thus, a b g = A B G, &c. this being the case; in the triangles T A K and T a K we have T A = T a; the angle A T K = a T K; and T A K = T a K; wherefore the sides T K must be equal in both triangles, (by E 1. 26) consequently the point K, which is the *entering point* of the side A B, is also the *entering point* of the side a b: in the same manner it may be proved that

the point H is the *entering point* of the lines GB and gb; also that the point M is the *entering point* of both FG, and fg. *Q. E. D.*

A *mechanical demonstration* will also be obtained if we suppose the *original plane*, ENWX, to be moveable about the *entering line* EN, by hinges fixed at the points E and N. then if the said plane be let down till it coincides with the *plane of the picture* ENrf, the several lines and points drawn in one plane ENWX will coincide with the corresponding lines and points in the other plane ENrf: that is, the points A and a, B and b, &c. as well as the lines TF and Tf, HG and Hg, &c. will coincide together.

XXVIII. *Remarks on the second and third Operations.*

The *vanishing line* VL (fig. 7.) is supposed given, or assumed on the *picture*; as is the point S for the *center of the vanishing line*. The perpendicular SI is then drawn on the *vanishing plane* VORL, and in this line the point I is taken for the *place of the eye* (Vide § 23.) Now if instead of drawing the line SI perpendicular to the *vanishing line* VL, in the *vanishing plane*; the line Si is drawn perpendicular to the same line VL, on the *picture plane*; and Si is made equal to (SI) the *distance of the eye*. Then the point i may be taken as the *place of the eye*; and will give the same

same *vanishing points*, as if the eye was supposed at I in the *vanishing plane* V O R L.

XXIX. As for example; to find the point d, which is the *vanishing point* of the *original lines* A B or a b, and of their parallels F G and f g; proceed as directed in the 3d operation; (§ 24.) by making the angle S i d = to the complement of the angle A T N or of its equal a T N; then the point d, where the line i d cuts the *vanishing line* V L, is the *vanishing point* of the line A B or a b.

D E M O N S T R A T I O N.

For the same thing being done in the *vanishing plane* V O R L, as was taught in the 3d operation; in the two triangles S i d and S I d, the side S I = S i and the angle d S i = d S I, and S I d = S i d whence the side S d is equal in both triangles (by E 1. 26.) and therefore the line i d, cuts the *vanishing line* V L, in the same point d, in which it is cut by the line I d. Q.E.D.

XXX. Having found the *vanishing point* of the *original line* A B or a b; the *vanishing point* of the line B G or b g, is found by the same method as was taught when the eye was in its proper situation I, in the *vanishing plane* V O R L (*Vide* § 25.) viz. on the *plane of the picture* p q r s, make at the point i, the angle d i D equal to the angle A B G, or a b g which the *original lines* B G and A B, or b g and a b, make with each other; and the line

C 3

i D

i D by cutting the *vanishing line* V L, in the point D, gives the *vanishing point* sought.

The demonstration of this is exactly the same as that of the last process: for a little reflection will point out that the triangles d I D and d i D are equiangular, and have one side in each d i and d I, equal; wherefore they are equal in all respects, (by E. 1. 26.) consequently the line i D, must cut the *vanishing line* V L, in the same point D, in which it is cut by the line I D.

XXXI. By § 27, we found the *entering points* T, H, K, M, (fig. 7.) of the *original lines*; and by § 29. 30, we have found their *vanishing points* d and D, without having any recourse to the *original* or *vanishing planes*: the lines joining those points are the *visual lines*, viz. d M, d K, D H, D T, (by § 14.) and their intersections in the points a, b, g, f, give the *representation* of the original figure a b g f. (*Vide* § 26.)

T H E O R E M X.

XXXII. When the *place of the eye* (i fig. 7.) and the *original figure* (a b f g) are laid down on the same plane; (p q r f) the *distance* (i d) of any *original line*, (a b) is parallel to that *original line*, (a b.)

D E M O N S T R A T I O N.

For V L being parallel to E N (by § 6.) the angle E K d = angle K d L, (E. 1. 29.) but
the

the angle $idD = \text{angle } aKE$ by construction : (*Vide* § 29.) and if we add these equal angles together we have angle $aKd = idK$ (E. 1. Ax. 2.) and consequently the line id is parallel to aK (by E. 1. 27.)

Again, since id is parallel to ab and the angle $d i D$ is equal to abg by construction (*Vide* § 30.) bg is parallel to iD . (E. 1. 28.) QED.

C O R O L L A R Y.

XXXIII. Hence the point (d) where a line (id) drawn on the picture ($p q r f$) through the place of the eye, (i) parallel to any original line, (ab) cuts the vanishing line (VL); is the vanishing point of that line.

From the foregoing observations it appears we need not make use of any more planes than one, which is the plane of the picture: and the original and vanishing planes may for the future be omitted. I shall always suppose the paper on which the schemes are printed, to represent the plane of the picture; on which two parallel lines EN and VL are drawn for the entering and vanishing lines of the original plane: all the space of the paper below the entering line EN may be accounted as the original plane: and all the space above the vanishing line, VL , may be esteemed the vanishing plane: on the part allotted for the original plane, there must (at present) be drawn the original figure in its proper situation and distance from the entering line

line E N. *On the part allotted for the vanishing plane there is always given the point i for the place of the eye. A perpendicular i S let fall from which, on the vanishing line V L gives the center S and distance S i of the vanishing line.*

Note, *That in all the plates throughout this work*

E N, is the *entering line* of the original plane.

V L, and sometimes H L is its *vanishing line*,

I, is the *place of the eye*.

S, is the *center of the vanishing line*: and

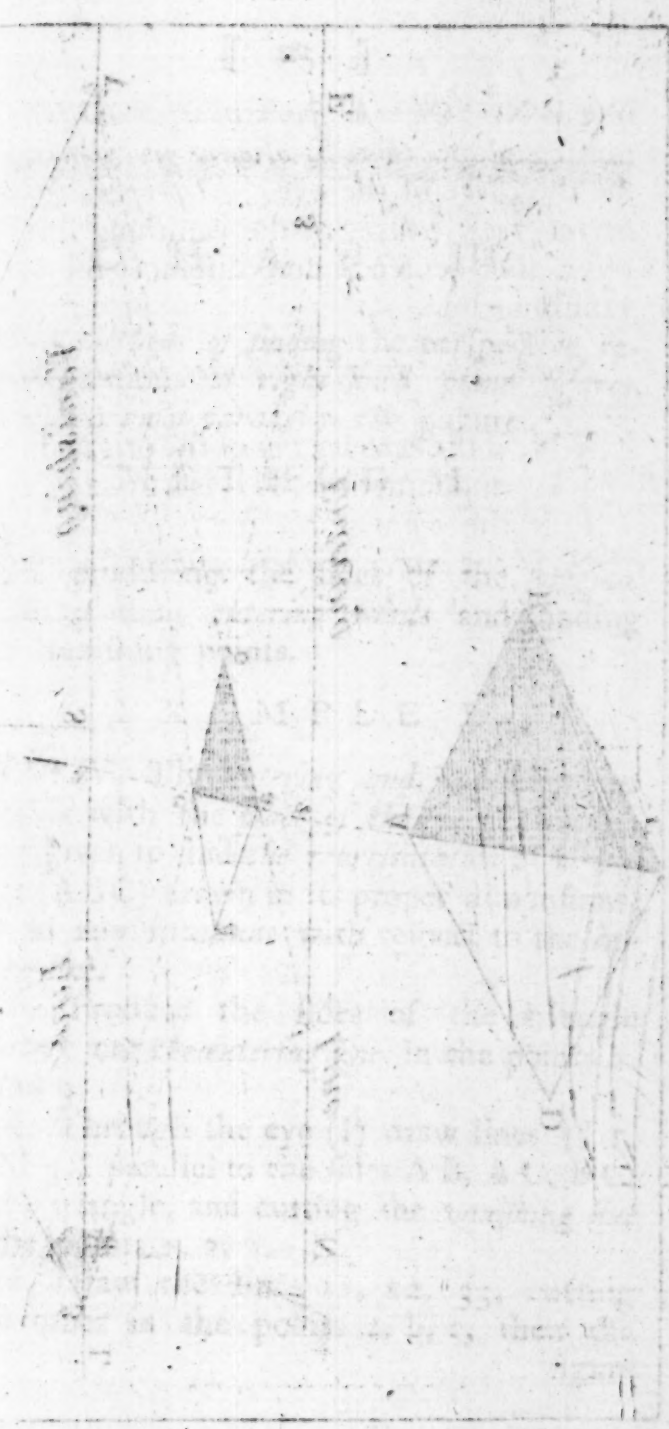
S I, is its *distance*,

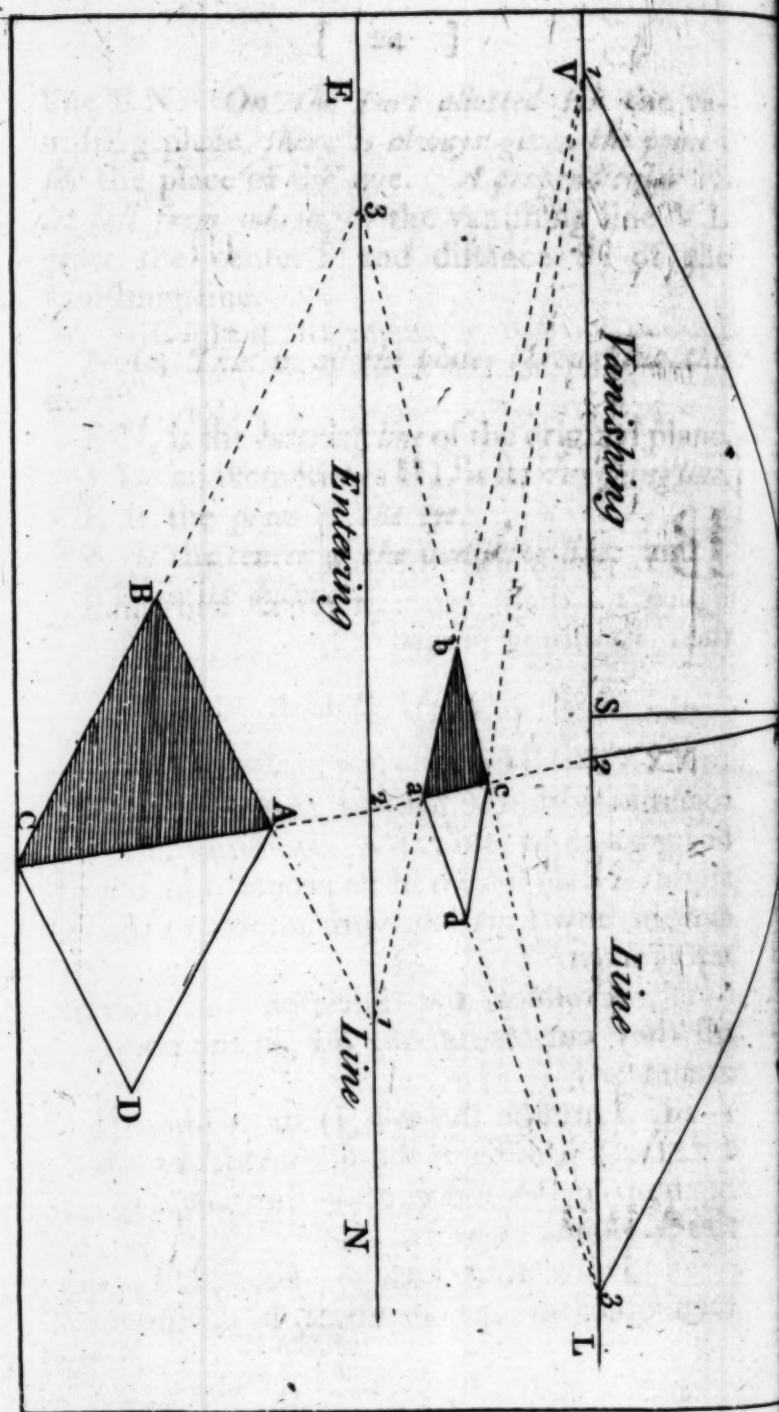
Vertical

Vertical

Vertical

Vertical







C H A P. III.

Various methods of finding the perspective representations of right-lined plane figures. which are not parallel to the picture.

M E T H O D I.

BY producing the sides of the *original* figure to their *entering points* and finding their *vanishing points*.

E X A M P L E I.

XXXIV. The *entering and vanishing lines* together with the *place of the eye* (I, fig. 8.) being given to find *the representation* of a triangle (ABC) drawn in its proper dimensions; and in any situation with respect to the *entering line*.

1st. Produce the sides of the triangle till they cut *the entering line* in the points 1. 2. and 3.

2d. Through the eye (I) draw lines (I 1. I 2. I 3.) parallel to the sides AB, AC, BC, of the triangle, and cutting the *vanishing line* in the points 1. 2. 3.

3d. Draw the lines 11, 22, 33, cutting each other in the points a, b, c, then the
figure

figure abc is the *perspective representation* of the original triangle ABC .

DEMONSTRATION.

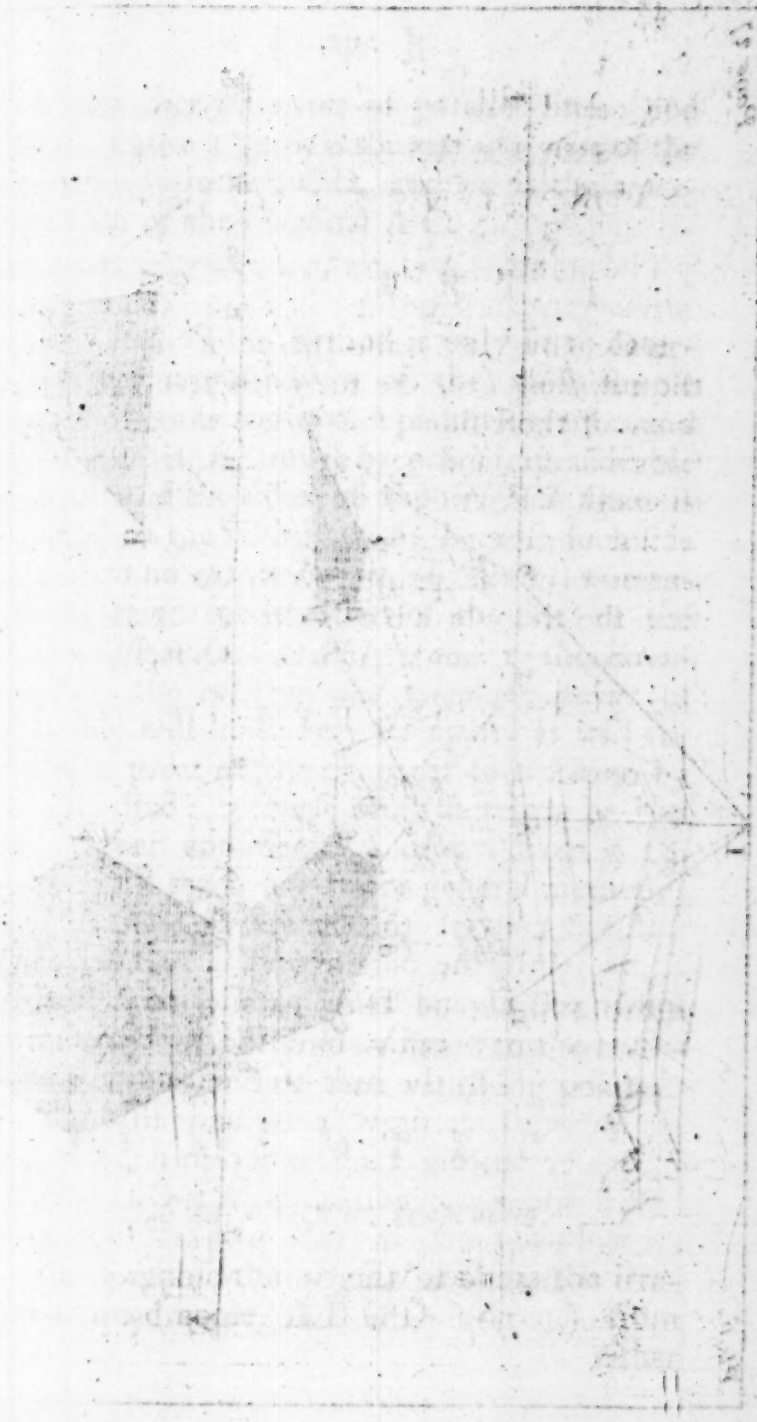
The points 1, 2, 3, in the *entering line*, are the *entering points* of the sides of the *original figure* ABC : (by § 27.) also the points 1. 2. 3. in the *vanishing line*, are the respective *vanishing points* of those sides (*Vide* § 33.) wherefore the lines 11, 22, and 33, are the *visual lines* of the sides AB , AC , BC . (§ 14.) whose intersections in the points a , b , c , give the *representations* of the points A , B , C , and therefore the triangle abc is the *representation* of the original triangle ABC . Q.E.D.

EXAMPLE II.

XXXV. *The representation of a parallelogram having all its sides oblique to the entering line: by means of its diagonal.* (fig. 8.)

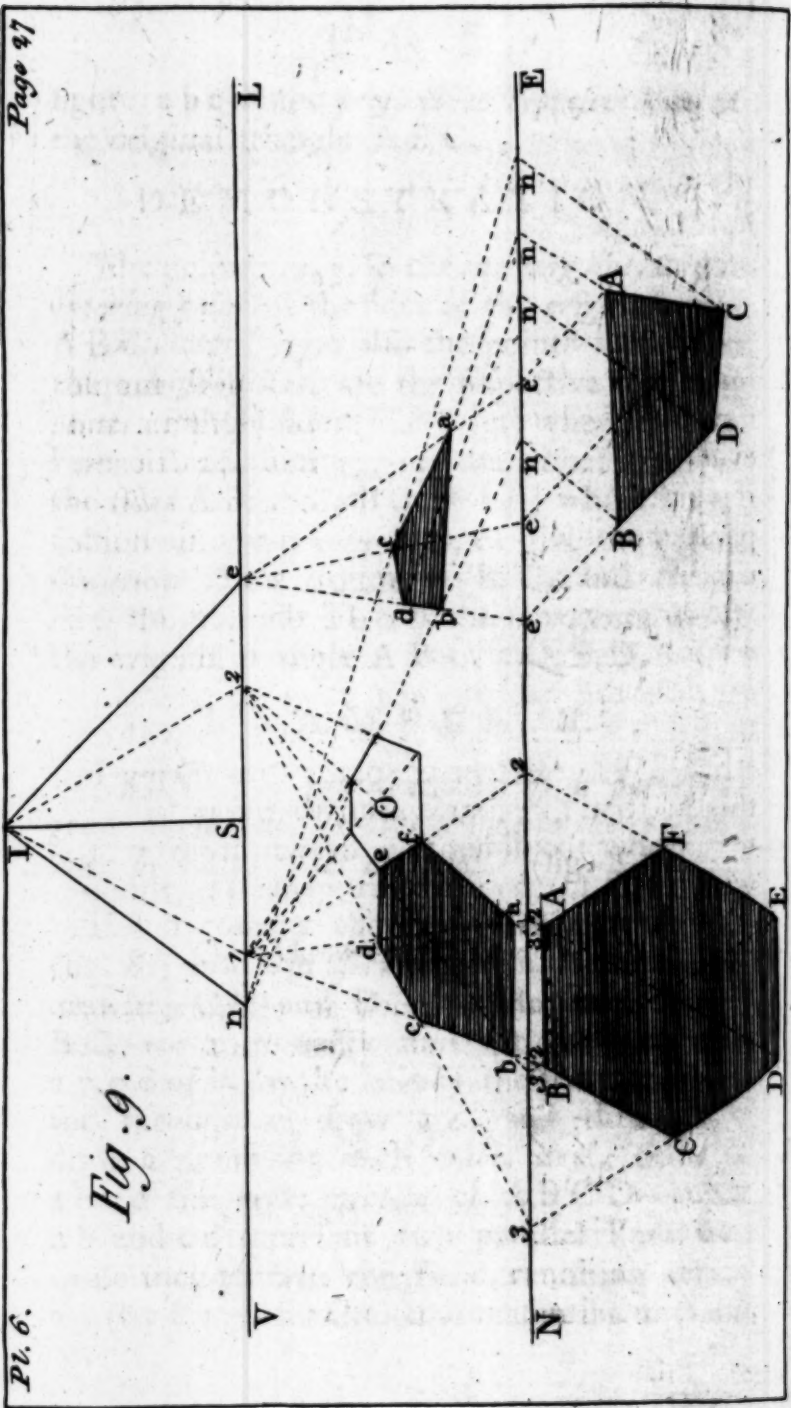
If we convert the *original triangle* ABC (fig. 8.) into the parallelogram $ABCD$, by drawing CD and DA parallel to AB and BC , we may easily find the *representation* $abcd$ of it, by the help of the triangle abc , for through c , draw $c1$; and through a draw $a3$, cutting each other in d ; then is $abcd$ the *representation* of $ABCD$ — For ab and cd represent two parallel lines because they tend to the same vanishing point 1; (by § 17.) for the same reason bc and ad are

12



11

Fig. 9.



are the *representations* of parallel lines; and so the figure $a b c d$, is the *representation* of the parallelogram $A B C D$, and $a c$ is the *representation* of the diagonal $A C$.

R E M A R K.

XXXVI. This method is only fit to determine the *representation* of the most simple figures; as a triangle, or a parallelogram: and even in these figures it becomes impracticable if either of the sides of the *original figure* is parallel to the *entering line*; because such side can have no *entering point*, (by § 11.) nor *vanishing point*: or if either of the sides of the *original figure* are nearly parallel to the *entering line*, the *entering and vanishing points* of that side will lie so very far apart, as will require a great extent of paper to work on by this method: recourse must therefore be had to another consideration drawn from § 18. which will teach us a more general method.

M E T H O D II.

By drawing two lines at pleasure from each angle of the *original figure* to the *entering line*; and finding their vanishing points.

E X A M P L E.

Of an irregular trapezium.

1st. Suppose the *original object* is the irregular quadrangle $A B C D$ (fig. 9.) from
either

either of its angular points, as from the point A, draw any two lines at pleasure as A e and A n cutting the *entering line* in e and n; then through the eye (I) draw the lines I e and I n, parallel to A e and A n, cutting the *vanishing line* in e and in n: join the points e, e and n, n by lines cutting each other in a, which is the *perspective representation* of the point A.

DEMONSTRATION.

The lines A e and A n may be considered as two *original lines*, whose *entering points* are the points e and n in the *entering line* E N: also since I e and I n are drawn through the eye parallel to these original lines A e and A n; the points e and n in the *vanishing line* are their *vanishing points*. (by § 33.) Therefore the lines e e and n n are the *visual lines* of the said originals A e and A n; (by § 14.) whose intersection in a, gives the *perspective representation* of the point A, which is the intersection of the *original lines* A e and A n (by § 18.) Q E D.

In like manner the *representations* of the other angular points B, C, D, of the *original figure* may be found: and these points being joined will give the figure a b c d; which is the *perspective representation* of A B C D.

XXXVII. The lines A e and A n were drawn at pleasure in order to determine the *representation* a, of the point A; but to find the

the *representations* of the other angular points as B, D, and C, the best way is to draw two lines from each of these points, parallel to those already drawn from the point A; for then the points e and n in the vanishing line, which are the vanishing points of the lines A e and A n; will be the vanishing points of those lines which are drawn from the other points B, C and D. (by § 10.)

E X A M P L E.

Of a regular hexagon.

XXXVIII. A consideration of the nature and properties of the figure to be represented in *perspective*, and of its situation with respect to the *entering line*, will often dictate modes of abbreviating the general rules of working. —Thus, if the *original figure* to be represented is a regular hexagon as A B C D E F (fig. 9.) a learner attentive to the properties and situation of this figure, will not find the *representation* of the point F by drawing lines from it to the *entering line* at random, as taught in this general method; because he will save much trouble by producing the sides E F and F A until they cut the *entering line* in the points 2, and 3, and finding their *vanishing points* by the parallels I 1, I 2; the *representation* of the whole figure may easily be found thus; join A and D, and produce B C, cutting the *entering line* in the points 2 and 2; also

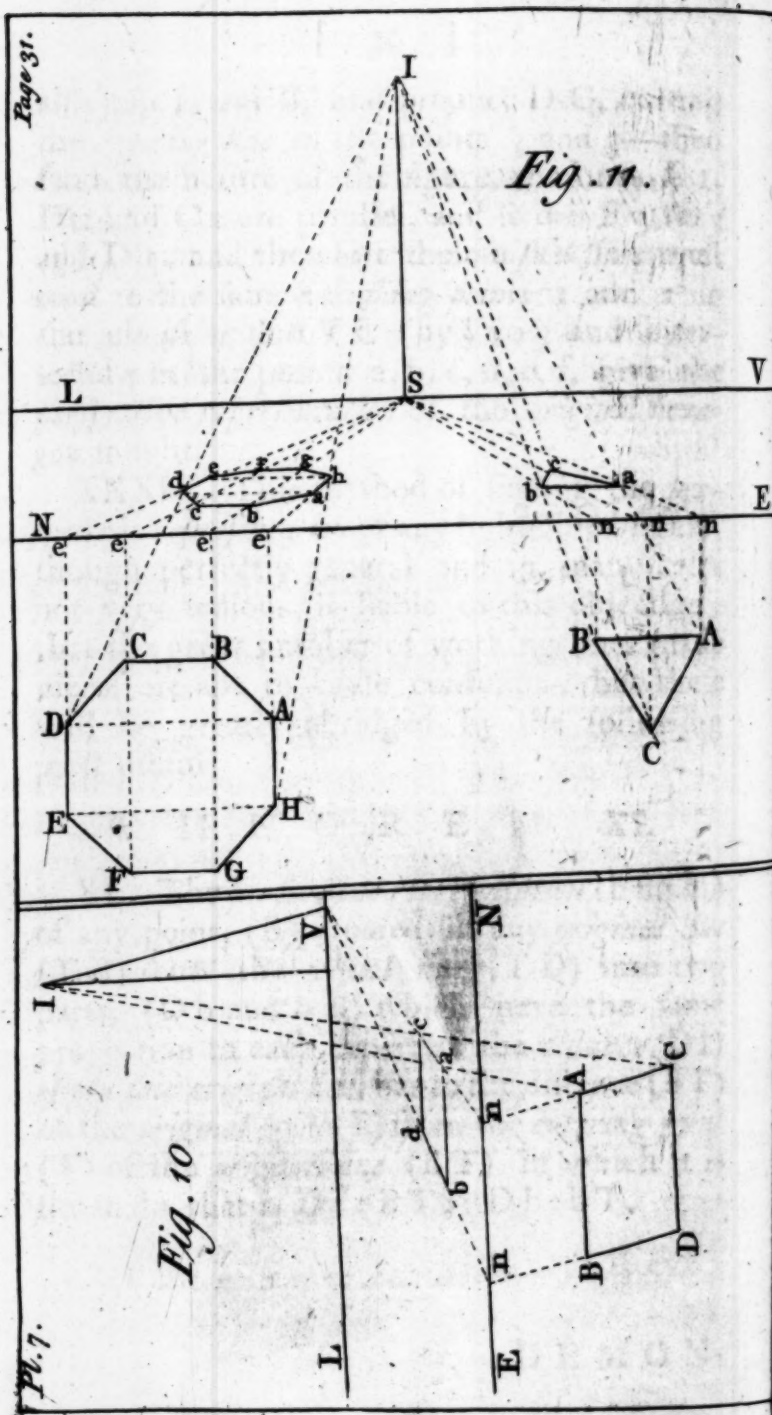
also join E and B, and produce D C, cutting the *entering line* in the points 3 and 3—then from the nature of the figure, the lines E 2, D 2 and C 2 are parallel, and so are F 3, E 3 and D 3. and therefore their *visual lines* must tend to the same *vanishing points* 1 and 2 in the *vanishing line* V L (by § 10.) and intersecting in the points a, b, c, d, e, f, give the *perspective representation* of the original hexagon sought.

XXXIX. This method of finding the *perspective representation* of any right-lined figure, though perfectly general and in many cases not very tedious, is liable to this objection; that the great number of working lines made use of are apt to cause confusion: but these will be greatly abridged by the following most useful

T H E O R E M XI.

XL. The *perspective representation* (b fig. 2.) of any point, (B) situated in any *original line* (T B) cuts the *visual line* (T D) into two parts, (D b and b T) which have the same proportion to each other, as the *distance* (D I) of the said *original line*, has to the distance (B T) of the *original point* (B) from the *entering point* (T) of the *original line* (B T) in which it is situated: that is $DI : BT :: Db : bT$.

D E M O N -



DEMONSTRATION.

The triangles $D b I$ and $b B T$ are both situated in the *visual plane* $ASIH$ (by § 12. and 16.) also since DI is parallel to BT (by § 9.) the angle $DIb = bBT$ (by E. 1. 29.) and the angle $bDI = bTB$, and so the triangles $D b I$ and $T b B$ are similar wherefore (by E. 6. 4.) $DI : TB :: Db : bT$. Q. E. D.

C O R O L L A R Y.

XLI. Hence, to find the representation, b , (fig. 2) of any *original point* B , we have nothing more to do than to cut the visual TD in such a manner that the parts Tb and bD may have the same proportion to each other as TB and DI have to each other, which may be done by any of the methods shewn in E. 6. 10. and from this consideration arises this

THIRD METHOD,

Of finding the perspective representation of a given right-lined plane figure, not parallel to the picture; by drawing lines from the several angles of the given figure to the place of the eye.

E X A M P L E.

XLII. *Of a parallelogram as $ABCD$ fig.*
10.

1 ft.

1st. Produce those two parallel sides CA and DB which will cut the *entering line* EN in the most direct manner, to their *entering points*, n and n.

2d. Through I draw IV parallel to the said sides AC and BD which gives the point V for their *vanishing point*. (by § 33.)

3dly. Draw the *visual lines* Vn and Vn (*Vide* § 14.)

4thly. Lastly, draw lines IA, IB, IC, ID, joining the eye and the angular points of the *original figure*, and cutting the said *visual lines* Vn and Vn in the points a, b, c, d, then the figure abcd bounded by the right lines joining these points, is the *perspective representation* of the *original figure* ABCD.

DEMONSTRATION.

For because IV is parallel to An, (by construction) the line AI cuts the visual nd in a, in such sort that $Va : na :: VI : An$. (*Vide* E. 6. 10. method first) wherefore a is the *perspective representation* of A by § 41.

In the same manner it may be proved that c is the *representation* of C; b of B; and d of D. Q. E. D.

E X A M P L E. II.

Of a triangle as ABC fig. 11.

XLIII. If it is inconvenient to produce the sides of the *original figure* to their *entering*

ing points, proceed as in § 37. viz. draw any lines at pleasure from the angular points of the *original figure* parallel to each other as A n, B n, C n, cutting the *entering line* in the points n, n, n—2d. Through the eye I, draw their *distance line* I S parallel to them, cutting the *vanishing line* V L in S which is their *vanishing point*. 3d. Draw the *visuals* n S, n S and n S. Lastly, draw lines I A, I B, I C, joining the eye and the angular points of the *original figure*; and cutting the *visuals* in the points a, b, and c; these when joined, form the triangle a b c which is the *perspective representation* of the *original triangle* A B C.

DEMONSTRATION.

If the line A n is considered as an *original line*, it is evident that a, is the *representation* of A, by the same argument as was used in the demonstration, § 42. also, b is the *representation* of B, and c of C.

EXAMPLE III.

Of a regular octagon.

The *representation* a b c d e f g h. (*Vide* fig. 11.) of the regular octagon A B C D E F G H is found exactly in the same manner as the triangle last mentioned; and needs no explanation; a bare view of the figure being sufficient to dictate its construction.

*XLIII. Although the parallel lines (Ae, Be &c.) drawn from the angles of the *original figure* to the *entering line* (E N) were directed to be drawn at random; yet in general it will be best to draw them perpendicular to the *entering line*, (as we have done in the two last examples) in which case (S) the *center of the vanishing line*, will be their *vanishing point*, and its *distance* will be their *distance*, by § 7. and 32.

L E M M A.

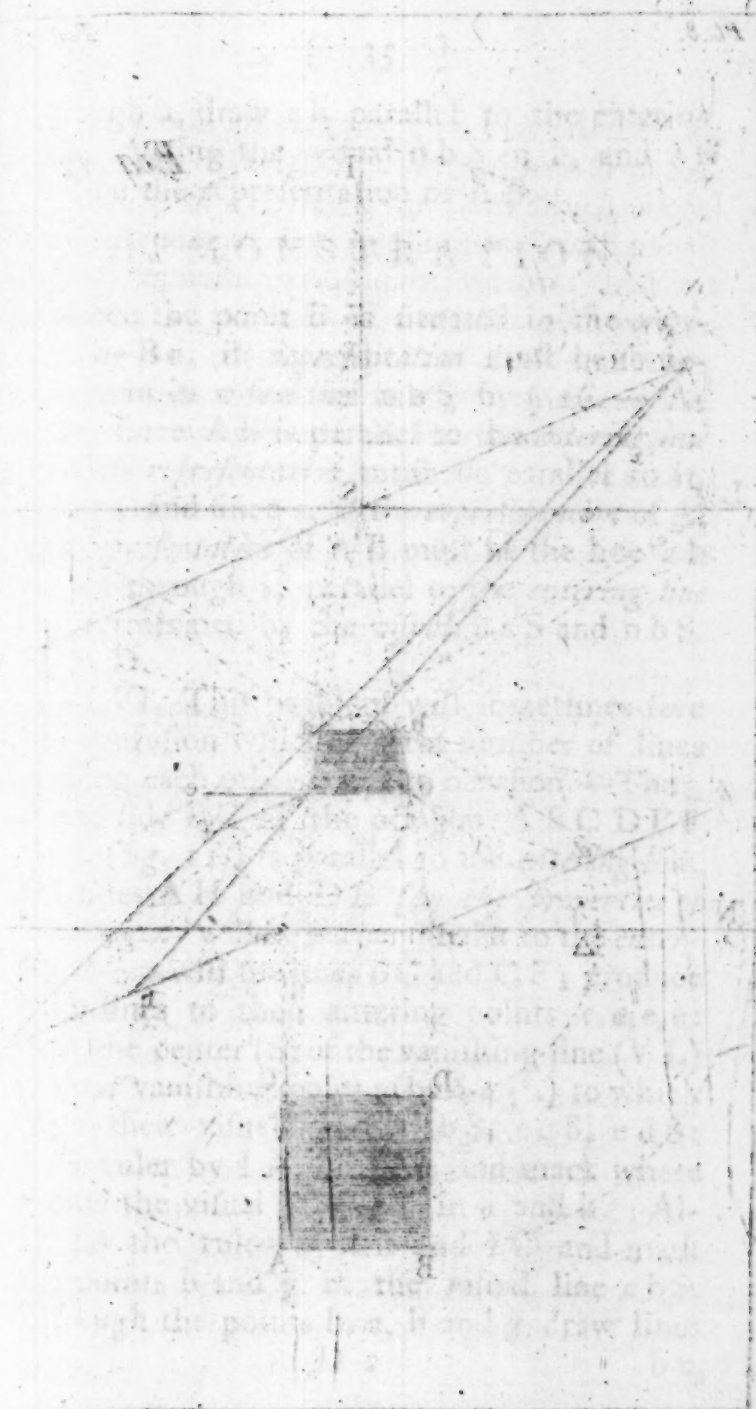
XLIV. If any *original line* is parallel to the *entering line*, its *perspective representation* will be parallel to the *entering line*.

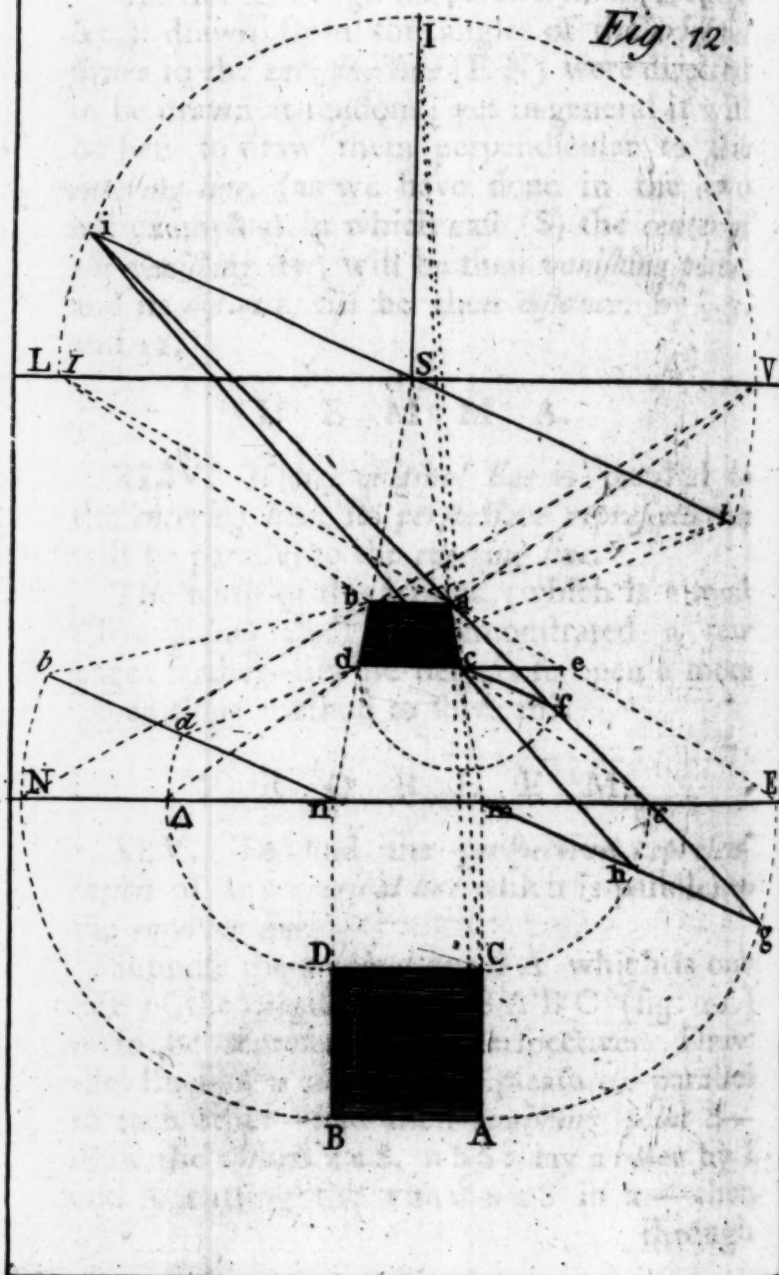
The truth of this lemma (which is almost self-evident) shall be demonstrated a few pages farther—its use here is to open a more expeditious method to solve this

P R O B L E M.

XLV. To find the *perspective representation* of any *original line* which is parallel to the *entering line*.

Suppose the *original line* B A which is one side of the original triangle A B C (fig. 11.) is to be represented in perspective. Draw the lines A n and B n at pleasure, parallel to each other—find their *vanishing point* S—draw the *visuals* n a S, n b S: lay a ruler by I and A cutting the visual n a S in a—then
through





through a, draw a b parallel to the entering line, cutting the visual n b S in b, and a b will be the representation of A B.

DEMONSTRATION.

Since the point B is situated in the *original line* B n, its *representation* must be somewhere in its *visual line* n b S by § 16. — Again, since AB is parallel to the *entering line* E N its *representation* must be parallel to it, by § 44. and since a, is the *representation* of A, the *representation* of AB must be the line a b drawn through a, parallel to the *entering line* and terminated by the *visuals* n a S and n b S. Q. E. D.

XLVI. This problem will sometimes save the confusion which a great number of lines crossing each other is apt to occasion.—Thus, if one side BC of the octagon ABCDEFGH (fig. 11.) is parallel to the *entering line*, the sides AH and DE (*by the properties of the octagon*) will be perpendicular to the *entering line*, as will the lines BG and CF; produce these lines to their entering points e, e, e, e: then the center (S) of the vanishing line (VL) is their vanishing point: (by § 43*.) to which draw their visuals eaS, ebS, ecS, edS: lay a ruler by IA and IH, and mark where it cuts the visual line eaS in a and h. Also, lay the ruler by IB and IG and mark the points b and g in the visual line ebS. Through the points b, a, h and g draw lines

D 2

bc.

bc , ad , he and gf , parallel to the *entering line*, and cutting the visuals ecS and edS in the points c , d , e , f : lastly, join ab , hg , fe and dc and the business is done—for the lines bc , ad , he , gf , drawn parallel to the *entering line*, are the *representations* of the *original lines* BC , AD , HE , GF , which by the nature and situation of the *original octagon* are parallel to the *entering line*.

Other methods may be proposed for finding the *perspective representation* of the angular points of any right-lined figure whose geometrical plan is given: which are derived from theorem XI, § 40. To explain these it will be necessary to prefix the following

D E F I N I T I O N.

XLVII. If through the extremities (n and S fig. 12.) of any *visual line* (nS) two lines are drawn at pleasure parallel to each other; (as nB and SI ; or nb and Si &c.) but tending different ways; that is, if one, as nB , tends downwards, its parallel SI may tend upwards; or, if one tends upwards and to the left, as nb , its parallel may tend downwards and to the right as Si &c. Then that line (nB or nb) which is drawn through the *entering point* (n) of the said *visual*, (*Vide* § 14.) is called a *scale line*. And the extremity (I , or i) of its parallel (SI or Si) which is drawn through the *vanishing point* (S) of the said *visual* (nS) and is made equal to its *distance* (SI)

(S I) is called the *dividing center* of that *visual line* (n S.)

FOURTH METHOD.

Of finding the perspective representation of a given right-lined plane figure, not parallel to the picture; by means of a scale-line drawn at pleasure, and a dividing center.

E X A M P L E.

XLVIII. A B C D (*Vide* fig. 12.) is a square, having two of its sides C D and A B parallel to the *entering line* E N: produce the other two sides to their *entering points* n and m. Through I draw I S parallel to them cutting the *vanishing line* in S which is their *vanishing point*. (by § 33.) Next, draw the *visuals* n S and m S—Now to find the *representation* b of any point, B, in any *original line*, as B n, nothing more is to be done than to cut its *visual* n S into two parts, n b and b S, so that $n b : b S :: B n : S I$. (*Vide* § 41.) wherefore draw at pleasure any two lines n b and S i parallel to each other, through the extremities n and S of the *visual line* n S; on the *scale line* n b (*Vide* § 47.) Lay from the *entering point* n, the several lengths n d and n b equal to the lengths n D and n B on the *original line* n B. Then on S i lay S i equal to I S (the distance of the *original line* D B). Lastly, draw the lines b i and d i joining the *dividing center*, and the several divisions b and d on

d on the scale line *n b* and cutting the visual *n S* in *b* and *d* which are the representations of *B* and *D* (by E. 6. 10. method 1st.)

M E T H O D V.

By using the entering line as a scale line, and laying the dividing center on the vanishing line.

XLIX. Since the *entering* and *vanishing lines* (*E N* and *V L*, fig. 12.) are parallel to each other (by § 6.) and pass through the extremities of all *visual lines* whatever (by § 14.) the *distance* (*S I*) of any *visual line* (*n S*) may be laid on the *vanishing line* (*V L*) from its *vanishing point* *S* to *V*, (then is *V* the *dividing center* of *n s* (by § 47.) and the *entering line* *E N* may be considered as a *scale line* for all *visual lines*, on which any part or parts *n D* and *n B* of the original line, (*n B*) belonging to that *visual* *n S*, may be laid from its *entering point* *n* to Δ and to *N*—then lines ΔV and *N V*, joining these divisions on the *scale line* (or *entering line*) *E N* and the *dividing center* (§ 47.) will cut the *visual* *n S* in the points *d* and *b*, the same as by the last method, and for the same reason. (E. 6. 10. method 1.)

Having, by any of these methods found the *representations* *b* and *d* of the *original points* *B* and *D*; through those points draw lines *b a* and *d c* parallel to the *entering line*, and cutting the *visual line* *m S*, in *a* and *c*,
then

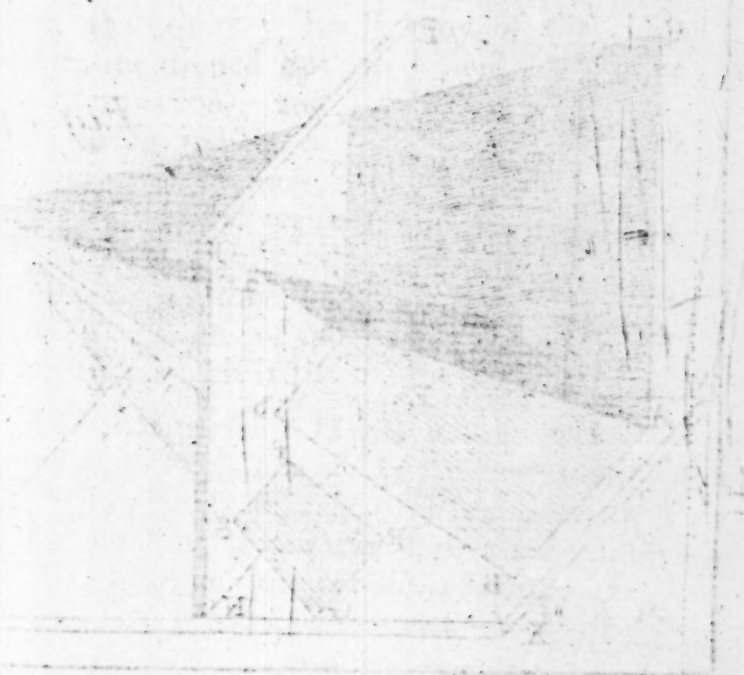
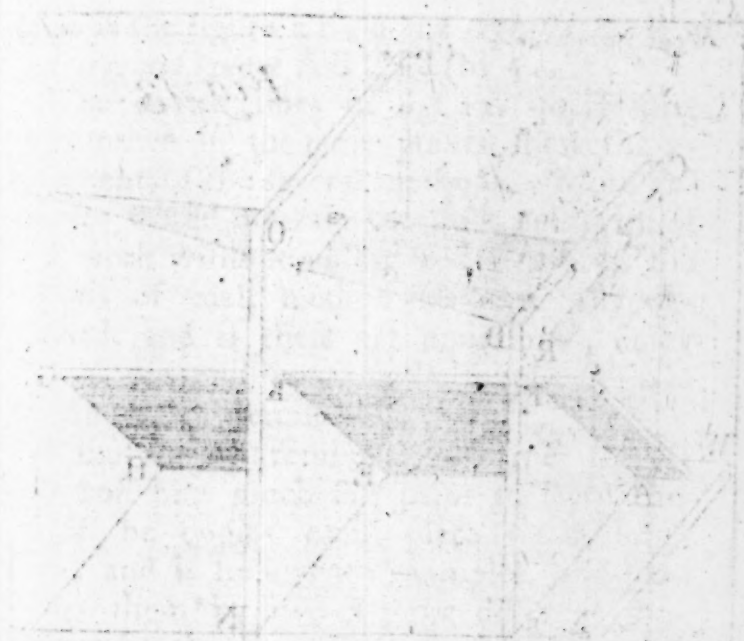


Fig. 13.

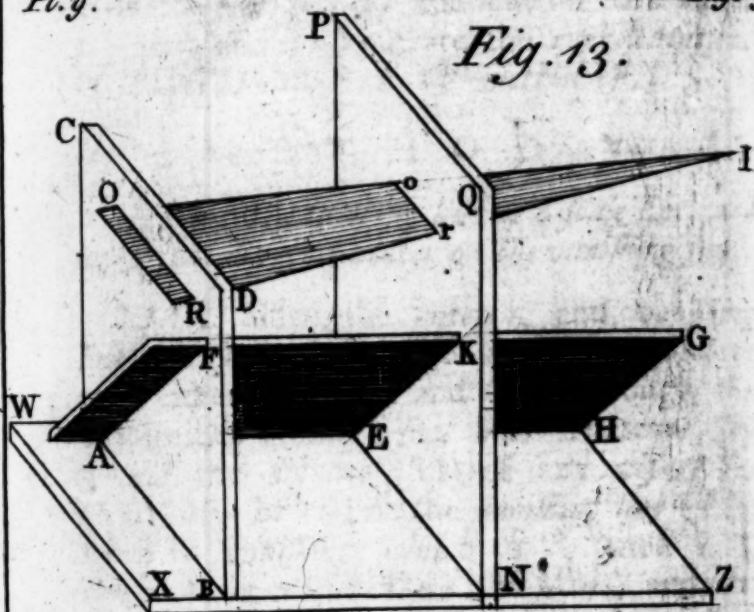
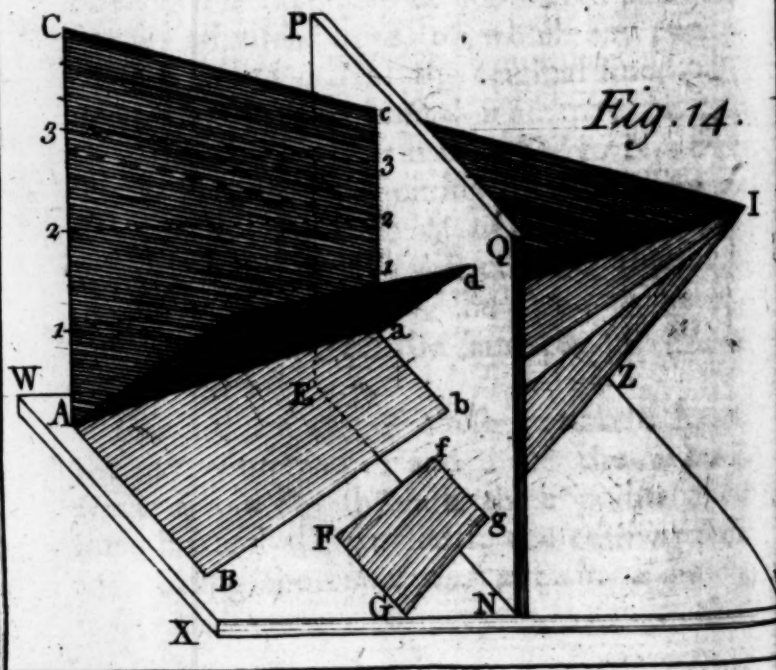


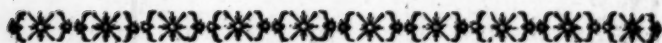
Fig. 14.



then is the figure $a b c d$ the *representation* of the *original square* $A B C D$ (by § 44.)

The dotted lines in fig. 12. intersecting each other in the same points, shew the agreement of the several methods. More examples would increase the bulk and price of this work without adding to its utility: the reasons of each method has been fully explained, and if these are understood, fancy and nature will continually supply subjects on which to exercise them.

I must again remind the learner that it will cost him much less pains to learn this art if he copies each plate on a large scale; and if he invents examples, and performs them by two or more different methods; in the execution of which he will find, that there is hardly any of the rules before-mentioned but what will sometimes be commodious, and sometimes inconvenient; being master of several ways of performing his business, he will rarely be embarrassed: if he wants room on his paper to work by one method, he will have recourse to another; not following any one particular mode of practice, but constantly using that which is the most advantageous.



C H A P. IV.

*Of the Perspective Representation of Lines
which are parallel to the Picture.*

P R O B L E M.

L. **B**Y any *original line* (A B fig. 13.) which is parallel to the *picture*; (P Q E N) to extend a plane (A B C D) which shall be parallel to the *picture*,

S O L U T I O N.

Suppose any plane as W X H Z to pass by the given line A B cutting the *picture* in the line E N. Through any point, E in, this line, draw at pleasure, on the plane of the *picture* the line E K; then pass a plane A F G H, at pleasure along the line E K, cutting the lines A B and E N in the points A and E. Through the point A draw on this plane A F G H the line A F parallel to E K (E 1. 31.) then a plane A B C D passing along the lines A B and A F will be parallel to the *plane of the picture*, P Q E N.

DEMON-

DEMONSTRATION.

For the line AF is parallel to the line EK (by *construction*) also the line AB is parallel to EN (by *supposition*) wherefore the plane $ABCD$ passing by these lines AB and AF must be parallel to the picture which passes by the lines EN and EK , (by E. 11. 15.) Q. E. D.

COROLLARY.

LI. Hence; any *original line* as AB (fig. 13.) which is parallel to the *picture* may be supposed to be the section of any two planes $ABCD$ and $WXHZ$ one of which ($ABCD$) is parallel, and the other ($WXHZ$) is oblique to the *plane of the picture*, so that if produced it will cut it in its *entering line* EN .

THEOREM XII.

LII. The *perspective representation* (or , fig. 13.) of any *original line* (OR) which is parallel to the *picture* ($PQEN$); is parallel to that *original line* (OR).

DEMONSTRATION.

Since, by *supposition*, OR is parallel to the *picture*; a plane $ABCD$ may be extended by it which shall be parallel to the *picture* (by § 50.) and since the visual plane (ORI) (*Vide* § 12.) cuts the two parallel planes AB
CD

CD and PQEN, the sections OR and or, are parallel: (by E. 11. 16.) but by § 16. or, is the *perspective representation* of OR, consequently the original line OR, and its *representation* or, are parallel. Q. E. D.

T H E O R E M XIII.

LIII. The *representations* (ab, fg, fig. 14.) of all *original lines* (AB, FG) which are parallel to the *picture* and to each other; are parallel.

D E M O N S T R A T I O N.

fg is parallel to FG (by § 52.) also AB is parallel to FG (by supposition) therefore fg and AB are parallel (by E. 11. 9.). Again since fg is parallel to AB; and ab is also parallel to AB (by § 52.) therefore fg and ab are parallel (by E. 11. 9.) Q. E. D.

C O R O L L A R Y.

LIV. Hence the *representations* (ab and fg) of all *original lines*, (AB, FG) which are parallel to the *entering line* (EN) of any *original plane*, (WXYZ) are parallel to that *entering line*.

N. B. *This is the lemma premised § 44.*

T H E O R E M XIV.

LV. If any two *original lines* (AB and AD. fig. 14.) forming an angle, (BAD) are

are parallel to the *picture*; the angle BAD which they make with each other) is equal to the angle $b a d$ which their *representations* ($a b$ and $a d$) make with each other.

DEMONSTRATION.

For the lines AB and AD are respectively parallel to $a b$ and $a d$ (by § 52.) wherefore the angles BAD and $b a d$ are equal (by E. 11. 10). Q. E. D.

COROLLARY.

LVI. Hence, if AB is an *original line* parallel to the *entering line* of any original plane $WXYZ$ (fig. 14.) and if AC is another *original line*, cutting the former at right angles, and also is parallel to the *picture*; then shall the *representation* $a c$, of that *original line* AC , be perpendicular to the said *entering line* EN .

DEMONSTRATION.

For the angle BAC being a right angle $b a c$ is a right angle (by § 55.) but $a b$ (which is the *representation* of AB) is parallel to the *entering line* EN (by § 54.) therefore ca is perpendicular to it. (by E. 1. 29.) Q. E. D.

THEOREM XV.

LVII. If two *original lines* (AB and AC fig. 14.) meeting in a point, are parallel to the *picture*,

picture, their *representations* (*a b* and *a c*) will have the same proportion to each other, as the *original lines* themselves have: that is, as $AB:AC::ab:ac$.

DEMONSTRATION.

In the triangular *visual planes* *ABI* and *ACI*, since *a b* is parallel to *AB* and *a c* parallel to *AC* (by § 52.) we have

$IA:IA::ab:AB$ (by E. 6. 4.) and

$IA:IA::ac:AC$ whence

$ab:ac::AB:AC$. (E. 5. 11.)

COROLLARY.

LVIII. Hence if the *original lines* (as *AB* and *AD*) are equal, their *representations* (*a b* and *a d*) will be equal.

THEOREM XVI.

LIX. If any *original line* (*AC* fig. 14) parallel to the *picture*, is divided into equal parts (*A 1. 12. 23. &c.*) the *representations* (*a 1. 12. 23. &c.*) of those parts will be equal: also, if the *original line* (*AC*) is divided in any manner, the *representation* of its parts, will divide the *representation* *a c* of the whole line in a similar manner.

The demonstration of this is the same as of the last Theorem.

CHAP.



C H A P. V.

Some applications of the preceding theory to the perspective representation of solid bodies.

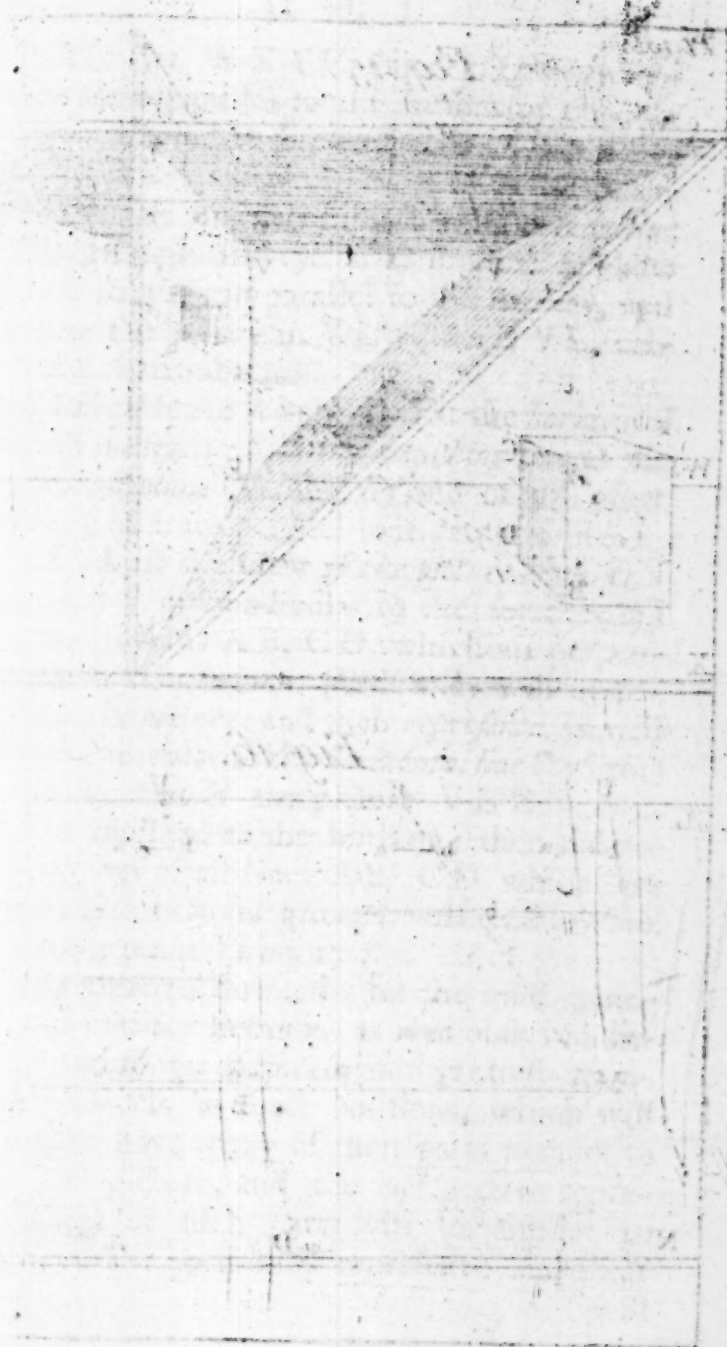
LX. **I**N explaining the *theory of perspective*, the *picture* and any *original plane* was considered in an abstracted manner, as two planes either parallel to, or cutting each other. But in applying our rules to the perspective representation of *natural objects*, the *plane of the Horizon*, and its parallels, claim particular attention; as all solid bodies must be supposed to stand upon such planes. For gravity (which draws every solid body downwards in a direction perpendicular to the horizon) imposes an erect situation on every thing that is to stand firm on its base: and although we may vary the ground plan of a building at pleasure, the walls will not stand without support, if they ascend in directions not perpendicular to the horizon.

For this reason builders work by the plumb-line and the square; that is, by lines perpendicular and parallel to the horizon. Architects, civil and military, delineate the dimensions of buildings and fortifications, by
sections

sections parallel to the horizon which they call *plans* *, and others perpendicular thereto, which they call *profiles* *. In fine, we cannot speak of the *length*, *breadth*, *height* or *depth* of any fixed solid magnitude, without referring to our ideas of an horizontal plane; by *length* and *breadth* we commonly mean its extent measured by lines parallel to the horizon; and by *height* and *depth* its altitude or profundity, estimated by the length of a line extending from the top or bottom of the object in question, in a direction perpendicular to the horizon.

What has been said will, I hope, be a sufficient apology for retaining the terms *horizontal plane* and *horizontal line*, without which we cannot treat this art in a practical manner; for although this *plane* and *line*, as Doctor Taylor observes, does not enjoy any particular advantage more than any other original plane and its vanishing line, in regard to the *theory of perspective*; yet gravity has made it and its parallels of so much consequence in the disposition of nature, that a picture can no more be drawn without a *horizontal line*, than a geographical map can be constructed without fixing the position of the first meridian.

* The words *Plan* and *Profile* are so generally understood, especially among artists, that a formal definition was esteemed unnecessary.



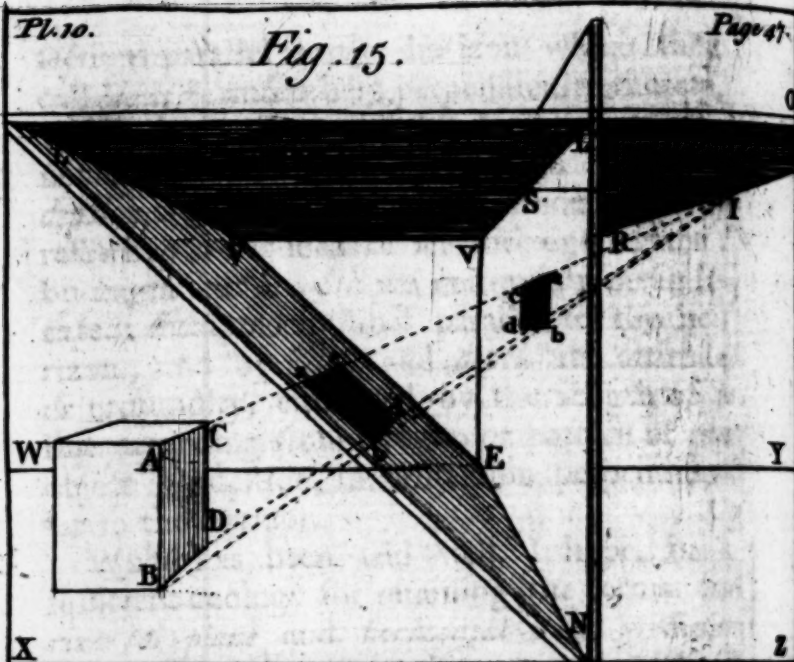
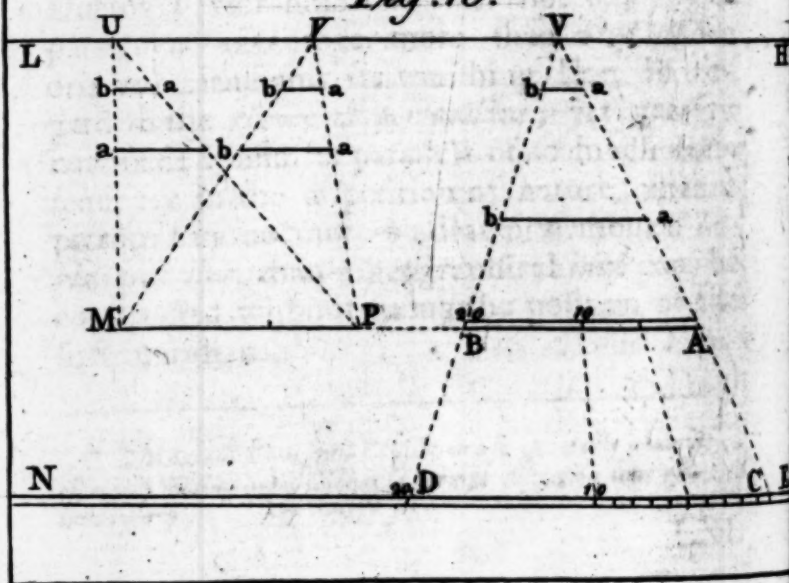


Fig. 16.



LXI. Let $WXYZ$ (fig. 15.) be any *original plane* parallel to the *horizon* or *ground*; and let $VLEN$, the *plane of the picture*, be either perpendicular or inclined to this *horizontal plane*, and cutting it in the *entering line* EN , through the eye at L , suppose a plane $VLO R$ to pass parallel to the *horizon*, and cutting the *picture* in VL ; then is VL , called the *horizontal line*.

LXII. Hence it appears that the *horizontal line* (VL fig. 15.) is the *vanishing line* of the *horizontal plane* ($WXYZ$) and of all other planes which are parallel to it. (by § 3 and 4.)

LXIII. If the *plane of the picture* ($VLEN$ fig. 15.) is perpendicular to the *horizon*; all original lines as AB , CD , which are perpendicular to the *horizon*. ($WXYZ$) will be parallel to the *picture*: and their *representation* will be perpendicular to the *horizontal line* (by § 56.)

LXIV. But if the *picture* $VLEN$ (fig. 15.) is inclined to the *horizon*; then the *representation* of all lines AB , CD which are perpendicular to the *ground*, will tend to their *vanishing point* (by § 17.)

The erect picture is by far the most general and useful; because, as was observed before, nature has disposed the greatest number of objects in erect positions, which will therefore have many of their parts parallel to the erect picture, and the perspective representations of such parts will be similar in shape to the parts they represent; and these will

will assist us to form a just idea of those parts of the original object, which by reason of their obliquity with the picture have their *representations* very unlike their real forms and proportions. Thus the *representation* $a b c d$ (fig. 15.) of the square $A B C D$ on the *upright picture*, is a square similar to the original, but on the *inclined picture*, it is an irregular trapezium totally unlike its original, $A B C D$.

The erect picture shall be first considered, on which the representation of any object may easily be drawn when the learner has made himself master of the two following problems.

P R O B L E M I.

LXV. The *entering* and *vanishing lines* ($E N$ and $H L$ fig. 16.) of any *original plane* being given; at any point in (A) in that plane, to put the *representation* ($A B$) of an *original line* which is parallel to the *entering line* ($E N$) and equal to any part ($C D$) of it.

1st. Through the given point A draw a line $A B$ parallel to the *entering line* $E N$;

2d. Through the same point draw any other line $V A C$ cutting the *entering* and *vanishing lines* in any two points as C and V .

3d. Lay from the point C on the *entering line* $E N$, the distance $C D$ equal to that part of the *original line* which is to be represented.

4th.

Fig. 17.

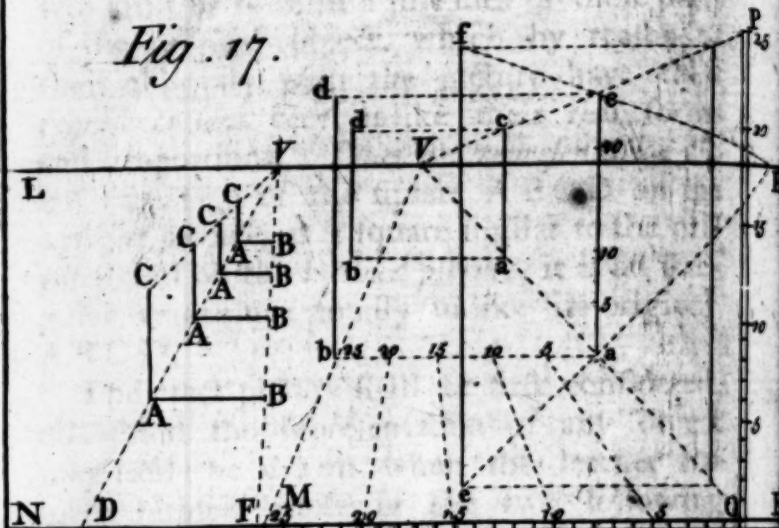
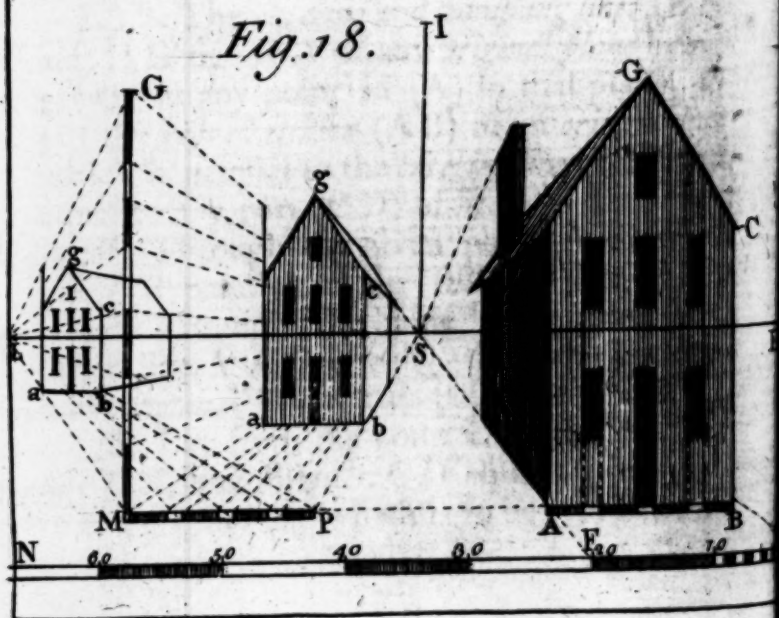


Fig. 18.



4th. Draw VD cutting AB in B , then shall AB be the *representation* required.

DEMONSTRATION.

For since AB , is parallel to EN , it is the representation of a line parallel to the entering line. (*Vide* § 54.) also CV and DV , are the representations of parallel lines, situated in that original plane whose entering and vanishing lines are given: (by § 17.) wherefore the figure $ABCD$ is the representation of a parallelogram, of which CD and AB being opposite sides, they must represent lines equal to each other: (E 1. 34.) consequently AB is the representation of a line parallel to the entering line, and equal to CD . Q. E. I.

COROLLARY.

LXVI. The *entering line* EN (fig. 16.) may be considered as a *scale*, from which any number of parts may be transferred to any line AB, drawn parallel to it. Thus if the part CD, of the *entering line* is 20 feet, then will AB be the *representation* of a line of 20 feet passing through A; and if this line is divided into 20 equal parts, each of them will represent one foot. (by § 59.) also the line AB thus divided, will serve as a scale to measure any other part of the same line produced. For if the whole length AB, is laid from M to P, in the same line AB produced; then will MP represent a line of 20 feet :
E because

because A B produced, is the *representation* of a line parallel to the picture; the equal parts of which, A B and M P, are the representations of equal parts. (*Vide* § 59.)

This measure, or any other, may again be transferred from M P, to any other line (a b), which is parallel to it, by the same means as A B was transferred from the entering line in § 65.

C O R O L L A R Y.

LXVII. The *representation* A B, of any line parallel to the *picture* being given; its dimensions C D, on the *entering line* may be found thus: through any point V, in the *vanishing line*, draw lines through the extremities A and B of the given line; cutting the entering line in the points C and D: then is C D the true measure of A B.

P R O B L E M II.

LXVIII. The *entering* and *vanishing line*, E N and H L, (fig. 17.) of any *original plane* being given: at any point A; (which is the *representation* of a point situated in the said original plane) to draw the *representation* A C of an *original line*, which is parallel to the picture, equal to any part (as D F) of the *entering line*, and inclined to the *entering line* or any of its parallels in a given angle.

S O L U T I O N.

1st. At the given point A, put the *representation* A B, of a line parallel to the *entering line*, and of the dimensions proposed; as suppose equal to D F (by prob. 1. § 65.)

2d. Through the same point A, draw A C, making with A B, the angle C A B, equal to the angle required: (suppose a right angle: Vide E. 1. 23. or 11.)

3d. Make A C equal to A B, and the problem is solved.

D E M O N S T R A T I O N.

For the two *original lines* represented by A B and A C, meet in a point represented by A, and are parallel to the picture; (by supposition) wherefore these *original lines* are inclined to each other, in an angle equal to C A B. (by § 55.) also because A B is equal to A C; it represents a line equal to that represented by A B. (by § 58.) but A B is the *representation* of a line equal to D F: (by construction) and so A C must also represent a line equal to D F.

This problem may be solved in another manner, which will often be more convenient. Thus suppose at the point a, (fig. 17.) I want to set the *representation* of a line which should be equal to any length proposed, and inclined to the *entering line*, (or to a line a b, drawn through the point a, parallel to the *entering line*) in a given angle.

1st. Through the given point *a*, draw any line (*V a O*) at pleasure; cutting the *entering line* in any point, as in *O*, and the *vanishing line* in any point as in *V*.

2d. Through *O* draw *OP*, making an angle (*NOP*) with the *entering line*, equal to the angle proposed (in this example a *right angle*.)

3d. Make *OP*, equal to the length of the line to be represented; taken from the scale on the *entering line*. (in this example 25 parts.)

4th. Through *a*, draw *ac* parallel to *OP*.

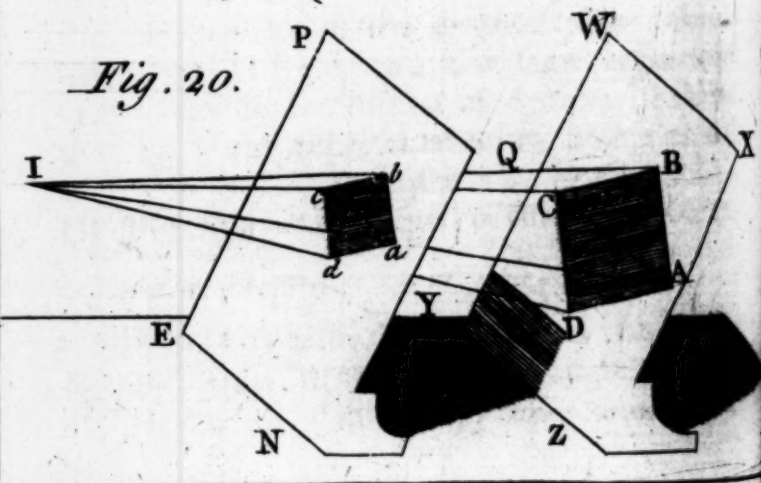
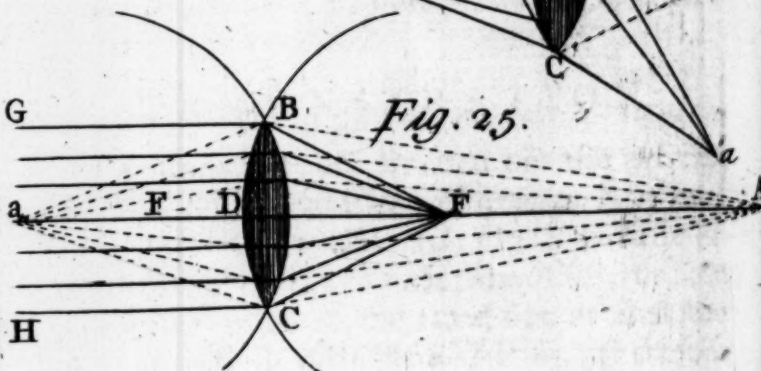
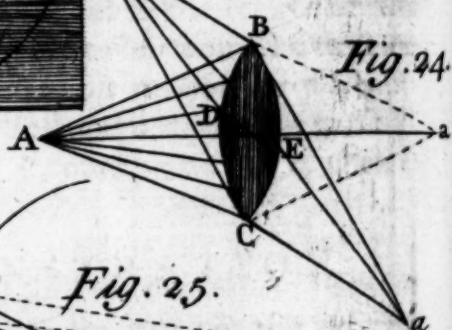
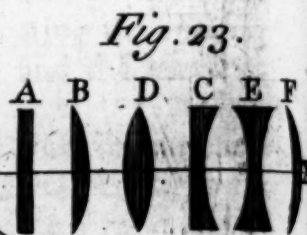
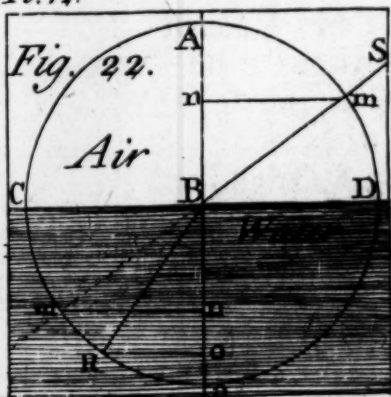
Lastly, draw *VP*, cutting *ac*, in *c*. then is *ac* the line required.

DEMONSTRATION.

For since *OP*, is situated on the *entering line* *EN*, and is parallel to the *picture*; it must coincide with it, (by E. 11. 1.) and the *representation* of the whole line *OP*, or any part of it, must be the same size as that line or part really is: the line *OP* is, therefore, the *entering line* of the *original plane* represented by *OPac*, and the same argument used in the demonstration of § 65. will shew that *ac*, is the representation of a line equal to *OP*. Moreover, since *ac*, is parallel to *OP*, the angle *cab*, is equal to the angle *NOP*.

Another demonstration may be had thus.

From *O* (fig. 17.) lay *OM* on the *entering line*, equal to *OP*, then through *a*, draw *ab* parallel to the *entering line*; and draw
MV



M V cutting a b in b: then is a b, the *representation* of a line parallel to the *entering line*, and equal to M O: (by prob. 1. § 65.) also by *similar triangles* (E. 6. 4.) we have

$$V O : V a :: O M : a b \text{ and}$$

$$V O : V a :: O P : a c \text{ whence}$$

(by E. 5. 11.) $O M : a b :: O P : a c$, consequently $O M : O P :: a b : a c$ (by E. 5. 16.) but as O M and O P are equal by construction; a b is equal to a c; and so this method is in effect the same as the former.

R E M A R K.

LXIX. It is evident that all the lines marked a c, drawn parallel to O P, and bounded by the *visuals* V O and V P; (see § 14.) are the *representations* of lines equal and parallel to O P: (by § 53. & 65.) and since O P contains 25 parts; if a c was divided into 25 parts, each of them would represent one of those parts.—a c being thus divided may be used as a *scale*, from which we may transfer the *representation* of a line parallel to the *picture*, and equal to any number of parts, to any other point as e: thus,

1st. Draw e f parallel to a c.

2d. Lay a ruler by e, and a, cutting the *vanishing line* in H.

3d. Draw H c, cutting e f, in f; then is e f, the *representation* of a line equal to a c. This is evident, because the figure e f c a, is the *representation* of a parallelogram.

T H E-

THEOREM XVII.

LXX. The *perspective representation*, (*a b c d* fig. 20.) of any *original figure*, (*A B C D*) situated in a plane (*W X Y Z*) parallel to the picture, (*P Q E N*) is similar to its original.

DEMONSTRATION.

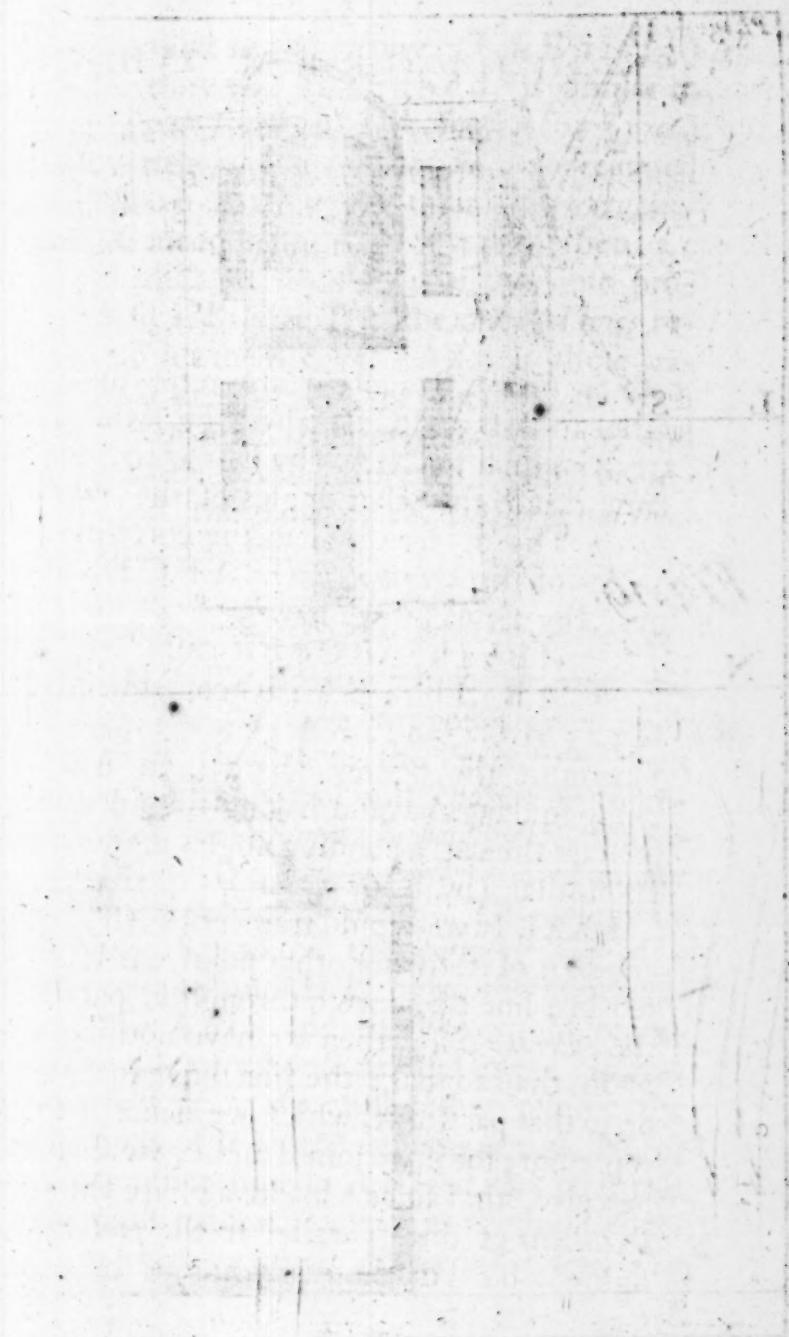
For since *a b* and *c b* are respectively parallel to *A B* and *C B*; (by § 52.) the angle *c b a* is equal to *C B A*. (by E. 11. 10.) for the same reason the other angles of the *representation a b c d*, are equal to the corresponding angles of the *original figure A B C D*, moreover, by E. 6. 4, we have

$$I B : I b :: B C : b c \text{ and}$$

$$I B : I b :: B A : b a \text{ consequently (by E. 5. 11.) } B C : b c :: B A : b a.$$

wherefore the figures *a b c d*, and *A B C D*, being equiangular, and having the sides which contain the corresponding angles proportional, are similar (by E. 6. def. 1.)

LXXI. If we would transfer *a c*, (fig. 17.) or any part of it, to any other point, (as *b*) situated in a line *a b*, drawn through *a*, parallel to the *entering line*; then we have nothing more to do, than to make the line *b d*, equal to *a c*; or to that part of it, which we desire it should represent; for if we join *d* and *c*, we shall soon perceive, that since *a b* and *a c*, are the *representations* of lines parallel to the *picture*, the figure *a b c d* is the *representation* of a *plane figure*



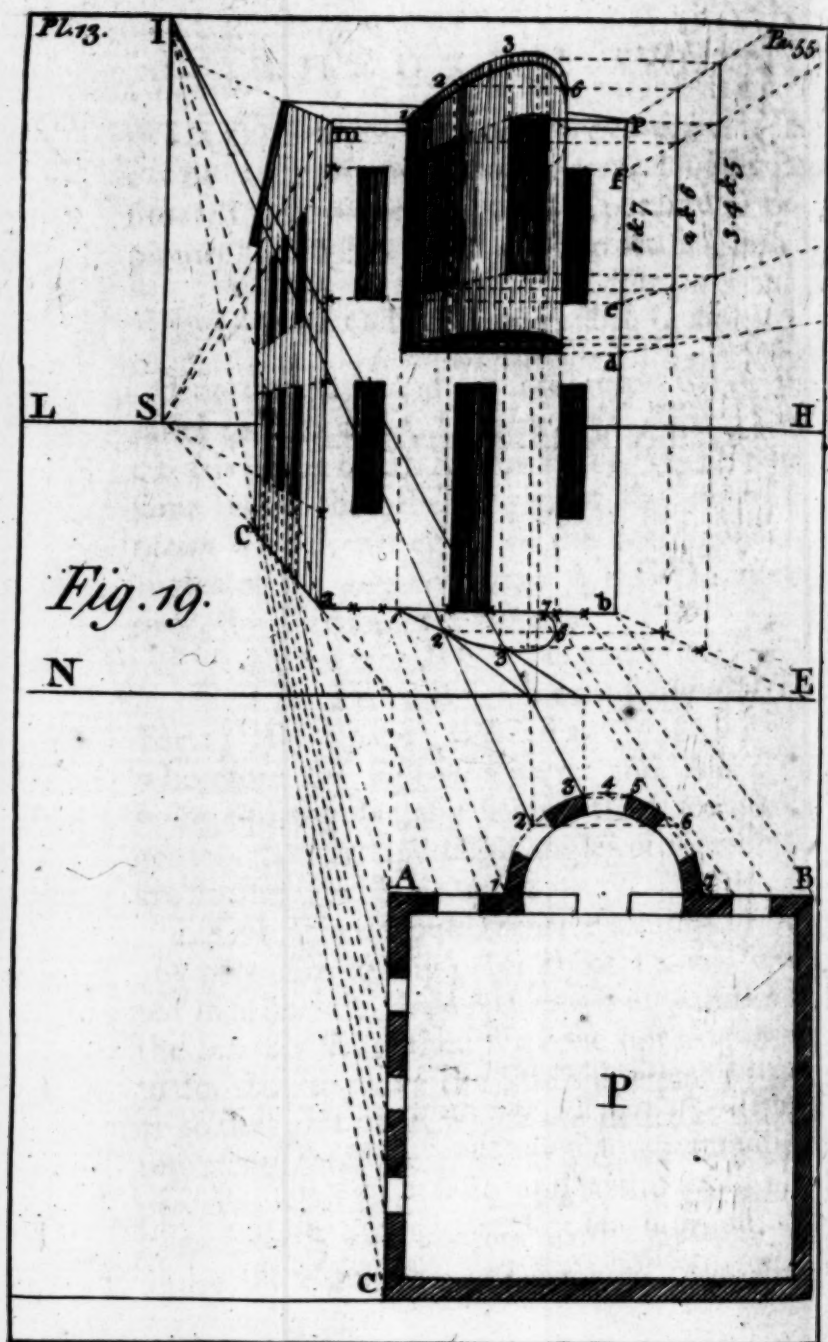


figure parallel to the picture. (Vide E 11. 15.) and therefore this figure *a b c d*, is similar to the *original figure* it represents. (by § 70.) consequently if the *original lines* represented by *a c* and *b d*, are equal, their *representations* must be made equal: if unequal, then *a c* and *b d* must be made to have the same proportion to each other, as the *original lines* represented by them have. Also, if those *original lines* are parallel, or inclined to each other; then must the *representations* *a c* and *b d* be made parallel to each other, or inclined just the same as the said *original lines* are supposed to be.

E X A M P L E. I.

LXXII. The plain whose *entering* and *vanishing line* we have supposed given, may be a plane parallel to the *horizon*: Then the two last problems will enable us to represent any *plane surface in perspective*, which is parallel to the *picture*, and situated on any part of the *horizontal plane*. Thus, suppose *EN* (fig. 18.) to be the *entering line* of a level ground; and *HL* to be the *horizontal line*; (Vide § 61.) and that we would represent the side of a house parallel to the *picture*: in the first place we must either assume the dimensions of it on the *entering line* *EN*, or on some other line as *AB* parallel to it: It is the most natural, as well as the most common method, to divide the *entering line*, into

a scale of equal parts; which may represent inches, feet, yards, &c. and this *scale* will be the true measure of all *original figures* which coincide with the *picture*. Suppose EF to be 20 foot, this is transferred to any line parallel to it, as AB, by problem 1. § 65, AB being divided into 20 parts, will be a scale to construct all the parts of the front. We may transfer the breadths of the windows, &c. to any other part (MP) of the line AB produced; and transfer their heights to any perpendicular line as MG; and from these two lines we may again transfer the several breadths and heights, to any other part of the *picture*, (as shewn by the dotted lines; *Vide* 66, and 69.) and construct fronts (as a b c g, a b c g, &c.) similar to the first front A B C G.

The oblique fronts of the houses are drawn to fill up the plate, by rules with which the reader is at present supposed to be unacquainted; but which shall be explained a few pages hence.

E X A M P L E II.

LXXIII. In fig. 19. P is the *ground plan* of a house, placed in any situation at pleasure, with respect to the line EN; which is the *entering line* of a level plane on which the house is supposed to stand: HL is the *horizontal line*, S is its center, and SI its *distance*. The ground plan, or any part of it, may be put

in *perspective* by any of the methods described in chapter 3. But I have made use of the 3d method § 42. which in general is the best. a b, the *representation* of A B, is first found; Then the situations of the windows and door, are transferred from A B to a b, by laying a ruler by the place of the eye (I) and the several divisions on the line A B; which cutting a b, will give the situations of the windows on the line a b, as expressed by the dots on that line: perpendiculars to the *entering* and *horizontal lines*, drawn through the points a, b and c, will be the *representations* of the corners of the house. (*Vide* § 63.) on either of these corners as a m, the heights of the windows, &c. may be transferred: (by § 68.) and by the help of these, and the divisions on a b, the front parallel to the picture may be completed; except the bow window: the oblique front, is also completed by means of these heights on the corner a m; and the dots on a c, representing the situations of the windows in the oblique front; which are found in the same manner as those on a b; viz. by laying a ruler by I, and the several divisions on A C, in the plan P. which ruler will cut the line c b in the parts expressed by the dots.— It is almost needless to remind the reader, that as the oblique front A C, of the plan, is perpendicular to the *entering line*; its representation a c, as well as the *representations* of the tops and bottoms of the windows,

&c.

&c. tend to S, the center of the horizontal line. (*Vide* § 7. and 43*.)—The bow window is next to be described — chuse any number of points in its semicircular plan, as the points 1. 2. 3. 4. 5. 6. 7. then by any method in the 3d chapter, find the *representations* of those points on the ground plane, and draw a *regular curve* through these *representations* 1. 2. 3. 4. 5. 6. 7. which will be the representation of the *original semicircle*. Perpendiculars being drawn through these points and made equal to the heights (b d, b e, b f, and b p,) of the several parts of the bow window designed to be represented; (by Prob. 2. § 68.) will give other points, through which *regular curves* being drawn, will be the *representation* of those parts. A careful perusal of the figure will make its construction more intelligible than a multitude of words.

LXXIV. When the *horizontal line*, and the *entering line of the ground plane*, are properly proportioned according to the distance of the eye; (by the observations contained in the two next chapters.) The ground plan of the original figure, if it is intricate, consisting of several complex parts, will, when put in perspective, be too confused to ascertain the exact points, where perpendiculars are to be raised, for determining the heights of those parts. To remedy this inconvenience, let a line be drawn parallel to the horizontal line, at a good distance from the *entering line of the ground*

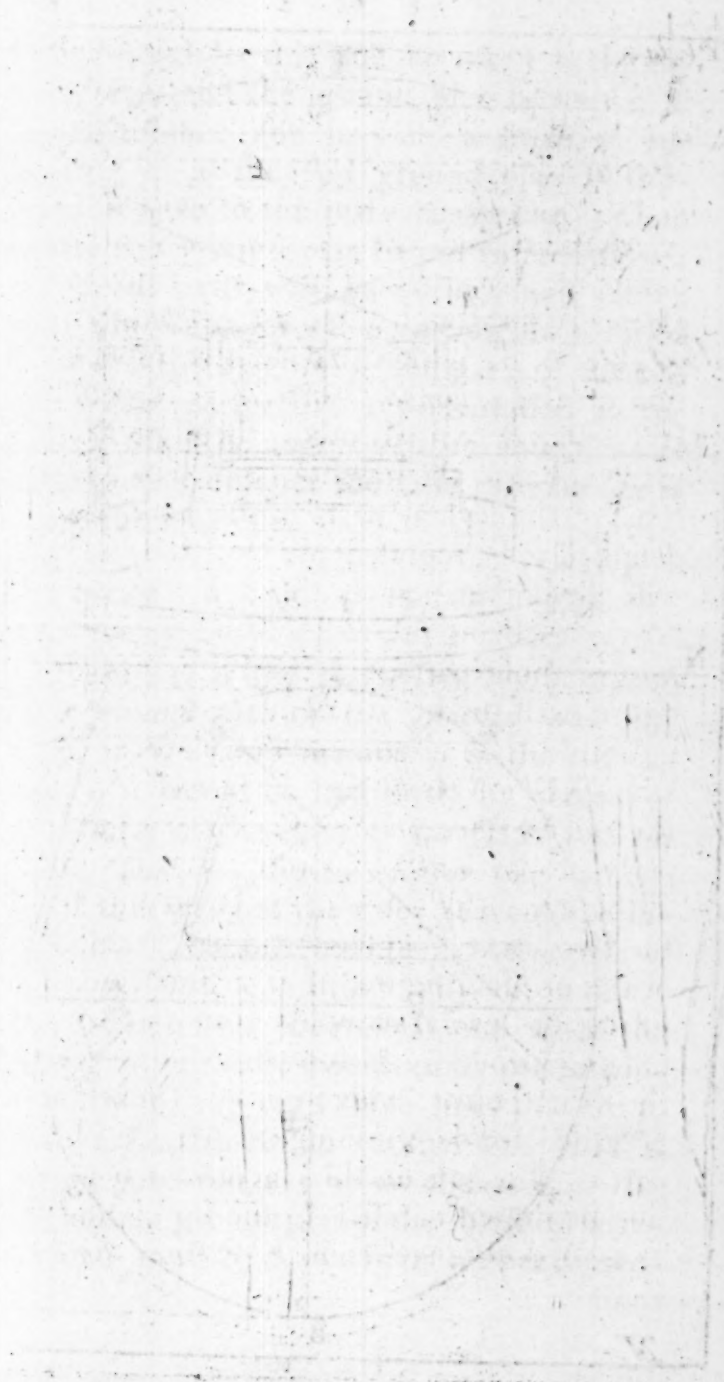
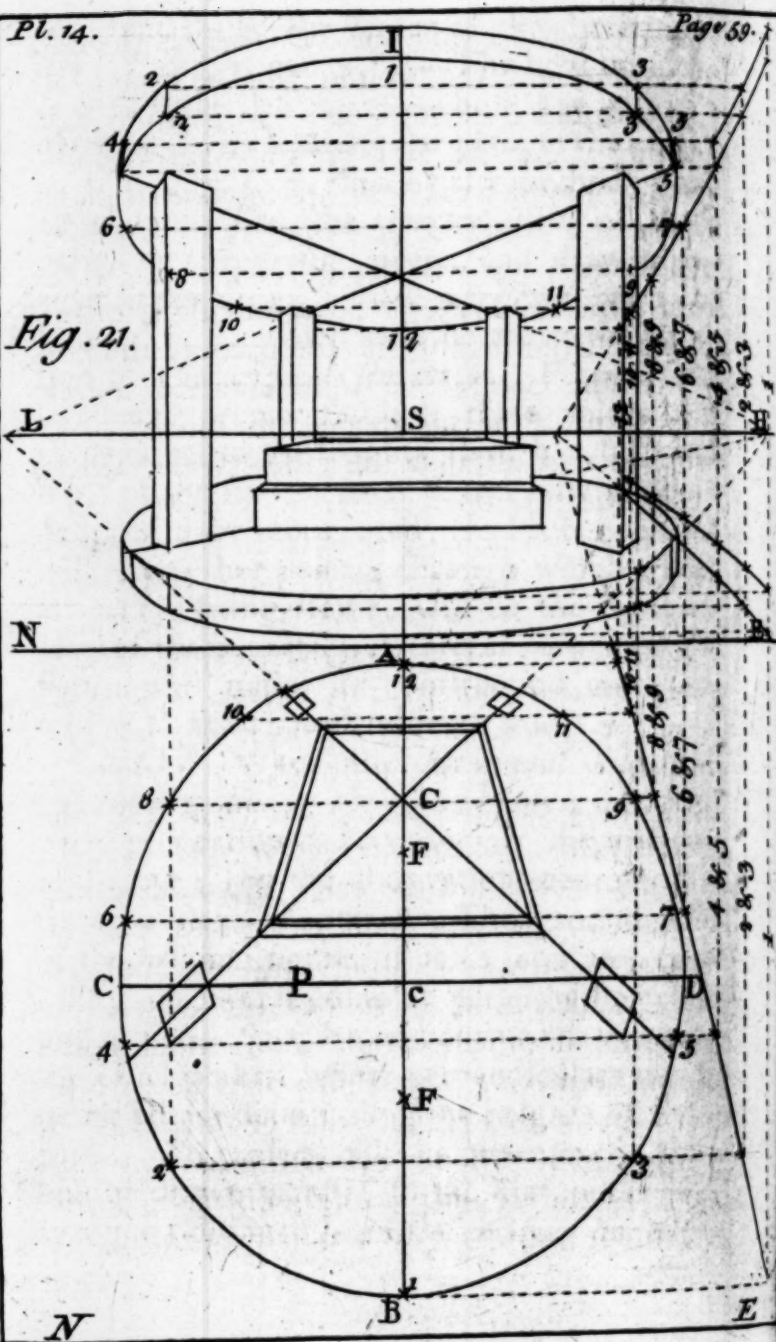


Fig. 21.



ground; and let this line be taken as the *entering line*, and the ground plan be drawn at such a distance and in such a situation respecting it, as the said ground plan is supposed to have to the real *entering line*. Then let the said ground plan be put in *perspective*; and all its parts will be distinct and clear: from which the several perpendiculars, raised to express the heights of the parts of the object whose perspective representation is required, must be augmented so much, as is equal to that distance the false *entering line* is below the true one.

E X A M P L E III.

In fig. 21. if the *perspective representation* of the *ground plan* of the objects was first found, in the same manner as in the last example; it would be indistinct, by reason of the vicinity of the *entering line* (EN), of the *ground plane*, to the *horizontal line* (HL). And if this was not the case, it would be inconvenient; because such a *representation of the ground plan*, consisting only of auxiliary lines, (which must be rubbed out when the drawing is finished) would cause much confusion, by mixing with those lines that are to remain: for this reason another line EN is drawn a good distance below the *entering line* EN ; and the ground plan of the objects to be represented, must be drawn in its proper dimensions

sions, and in the same situation with respect to EN , as it is supposed to have to the true *entering line* EN ; then esteeming the line EN as the true *entering line*; the *perspective representation* (P) of the said ground plan, is found by any of the methods taught in the third chapter. (N. B. The geometrical plan from which the *perspective representation* P is derived, is not inserted in the plate for want of room: it may also be remarked, that this *perspective representation* is much more distorted, but at the same time the intersections of the lines forming it, with each other, are much more distinct, than they would have been if we had made use of the true *entering line* EN . In chusing this false *entering line* EN , it should be taken at such a distance below the *real entering line*, EN , that the *perspective representation* of the whole ground plan, may fall entirely between these lines, as in the present example.) Having found the representation of the ground plan, the heights of the several parts are to be found, by drawing perpendiculars through their seats in the ground plan; and making those perpendiculars equal to the heights proposed, increased by the height EE or NN , that the true *entering line* EN is above the line EN . (*Vide* § 68. &c.) The reason of which is, because EN , is the *entering line* of a level plane, situated at the depth EE or NN below the plane of the ground; as will be evident on the

the

Fig. 26.

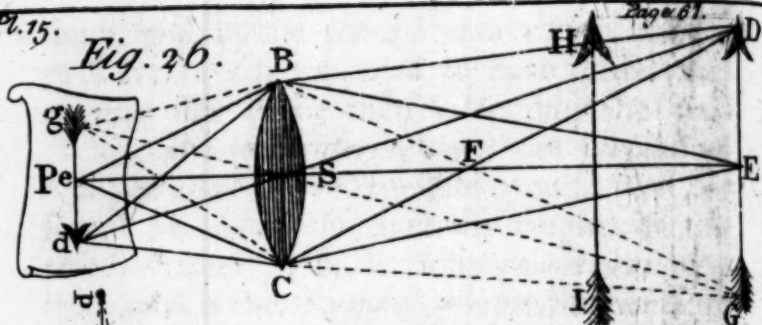


Fig. 28.



Fig. 27.



Fig. 29.



the least reflection. Every other step used for finding the *perspective representation* of the objects delineated on the plate, will appear by inspection, assisted with the skill which the learner is supposed to have acquired by the foregoing pages.

I forbear more examples, because the trouble of drawing a plan of every thing we want to represent in perspective is intollerable; and were there no other way of proceeding, our art would only supply the architect with a tedious method of drawing the *representation* of a building from its plan and profile: whilst the artist who is to dispose the *representations* on his picture in such a manner as to form an agreeable composition; could receive little or no assistance hereby: but we shall be able to surmount this difficulty, and not only abbreviate the labour, but also enlarge the utility of *perspective* by the methods taught in the seventh chapter.

CHAP.

C H A P. V.

A digression concerning the refraction of light, the construction of the eye, and the nature of vision.

BEFORE the nature of vision is explained, it will be necessary to speak of the change of direction which the rays of light suffer, when they pass through mediums of different forms and densities: this subject, in its full extent, constitutes the science which is called **DIOPTRICS**; on whose utility and excellence it were needless to expatiate, after observing, that to this science we owe the invention of those optical machines, which correct the imperfections of sight, and increase its natural powers; making us acquainted with beings, whose extreme minuteness render them imperceptible; and bringing to hand objects whose remote situation afford to the naked eye either no information, or such as is imperfect and erroneous.

1. *Rays of light issue from a luminous body as from the sun, a fire, a torch, &c. every way, in rightlined directions.* For if a torch is fixed in the middle of a large common, its rays are dispersed

perfed immediately all around: but none of thefe rays will pafs through the bore of a bended pipe. Consequently thefe rays iffue every way; and in *right-lined* directions.

2. Every thing that gives a paffage to light is called a *medium*: thus, air, water, glafs, diamond, &c. are mediums.

3. In the fame medium a ray of light proceeds in a right-lined direction, (as was obferved, Art. 1.) but when a ray paffes obliquely out of one medium into another of a different density, it is bent out of its *first direction*, into another *right-lined direction*. And this *inflection* or *bending*, is called *refraction*.

Example. The ray S B (fig. 22.) coming from the fun, through the air, ftrikes the furface of the water in the point B. it then receives a new direction B R, approaching nearer to the perpendicular B a. This *refraction* arifes, becaufe the ray S B is more ftrongly attracted by a dense medium, as water, than by a rare medium, as air.

4. The ray S B is called the *incident ray*, and the angle S B A which it makes with the perpendicular B A, is called the *angle of incidence*.

5. The part of the ray, B R, which is immerfed in the water, is called the *refracted ray*, and the angle R B a; which it makes with the perpendicular B a, is called, the *angle of refraction*.

6. If

6. If about the point B as a center, a circle, R A m , is described with any radius; (as B A) the arch A m or a m , measures the angle of incidence (A B S or a B m) and the arch a R , measures the angle of refraction (a B R):—But a line (m n) drawn through one extremity (m) of an arch, (A m) perpendicular to a radius (B A) passing through the other end (A) of the said arch (A m); is called the *sine* of that arch; wherefore m n , is the *sine* of the angle of incidence, and o R is the *sine* of the angle of refraction.

7. Arguments drawn from the principles of mechanics and geometry, demonstrate, and experiments confirm, that the *sine* (m n) of the angle of incidence, has always the same ratio to the *sine* (o R) of the angle of refraction, in whatever degree of obliquety the incident ray, strikes the surface (C D) which separates the mediums: this proportion is as follows:

When the incidental ray passes obliquely out of air into water; the *sine* of incidence (m n) is to the *sine* of refraction (o R) as 4 to 3, nearly.

If it passes out of air into glass, the proportion is as 3 to 2. And out of air into diamond, the *sine* of incidence, is to the *sine* of refraction as 5 to 2.

8. The opticians to answer their purposes grind the surface of glass into the different forms

forms exhibited in fig. 23 *. The mathematicians, from the general law of refraction, mentioned in the last article, and the curvature given to its surface, invest the properties of either of these glasses, which is called a *lens*. I shall mention one or two of their conclusions. Those who would see their arguments, may consult Mr. *Martin's mathematical institutions*, 8vo. vol. II. page 116. &c. or *professor 's Gravesands mathematical elements of nat. philos.* 4to. book V. chap. 8. &c. To these books, and to Mr. Fergusson's lectures, I am obliged for the heads of this chapter.

9. A line A X (fig. 23.) drawn perpendicular to both the faces of a lens, is called its *axis*. This line, if those faces are spherical, will pass through their centers.

10. A parcel of rays issuing from any *radiant point* (as A or A fig. 24.) and falling on a lens (B C) situated in any position with respect to that point, (A or A) is called a *pencil of rays*; of which that *single ray* (A a or A a) which passes through the center of the lens, is called, *the axis of that pencil*.

F

11. If

* A *Plane Glass* is flat on both sides as A.

A *Plano-convex*, is flat one side and convex on the other, as B.

A *Plano-concave*, is the reverse, as C.

A *Double-convex*, is convex on both sides, as D.

A *Double-concave*, is concave on both sides, as E.

A *Meniscus*, is convex on one side, and concave on the other, as F.

11. If a ray (as *A a* fig. 24.) strikes a lens (*BC*) in the *direction of its axis*; it will pass directly through it; because, as there is no *angle of incidence*, there can be no angle of *refraction*. (*Vide* art. 7.)

12. If a ray (*AD* fig. 24.) passes through the center (*D*) of a lens; but at the same time *strikes it obliquely*, it will be turned out of its way, in the *refracted direction* *DE*. (*Vide* art. 3.) But when it has passed through the lens (to the point *E*,) it will proceed in a *direction* (*E a*) parallel to the incidental ray (*AD*.) And if the thickness of the lens is inconsiderable (as is always the case,) the refracted part (*DE*) will make so small a difference; that the whole line *A a* may be esteemed a right line.

13. From hence it may be inferred, that the axis (*A a* or *A a* fig. 24.) of any pencil of rays passes through a lens (*BC*) in a right-lined direction. Because the *axes* of all *pencils*, pass through the *center of the lens*. (*Vide* art. 10.)

14. Rays proceeding from any *radiant point* (as *A* or *F* fig. 25.) are called *diverging rays*. And when they fall on a lens (*BC*) they are said to be *more* or *less diverging*, according as the *radiant point* (*A* or *F*) is *more*, or *less remote* from the lens. If the *radiant point* *A*, is very remote from the lens in respect of its diameter *BC*; (as if the *radiant point* is fifty yards distant from a lens, whose diameter is one inch.

"inch.) The *pencil* ABC, may be esteemed as composed of parallel lines.

15. It is the property of a *convex lens* BC, (fig. 25.) that if a *pencil of rays* GBHC, parallel to each other, fall on it in a direction parallel to the *axis of the lens*, a D. (*Vide* art. 9.) they will be so refracted, as to meet in some point (F) of that *axis* continued. This point, F, where these *parallel rays* meet, is called the *principal focus* of the *lens* BC. The distance of which, from the center of the *lens*, is called its *focal distance*.

16. In a *plano convex lens* the *focal distance* is equal to the *diameter of the convex side*: in a *lens equally convex on both sides*, the *focal distance* is equal to the *radius of the convexity of either surface*.

17. If a *radiant point*, F, (fig. 25.) is situated in the *focus* of a *convex lens*, BC, the *pencil of rays*, FBC, issuing from that point, F, will become parallel after passing through the lens *.

F 2

18. If

* Mathematicians in deriving these conclusions, only consider those *incidental rays*, which fall on the center of the *lens*, and are very little dispersed: because the rays which pass through different mediums that are separated by curved surfaces, are not all united by refraction: they consider different *pencils of rays* proceeding from the same *radiant point*, to insist on infinitely small and distinct portions of the surface which separates the mediums; from the *radius of the curvature* at any one of these points, and the constant proportion of the sines of *incidence* and *refraction*, they find a *focus*, where those rays will meet; then

18. If the *radiant point* A, is situated any where beyond the *focus* F, of the *convex lens* B C. (fig. 25.) the *pencil of rays* A B C, after passing through the *lens*, will converge to a point, a, in the *axis* (F D) of that *pencil* produced; which point a, will be nearer to, or more remote from the *lens*, according as the *radiant point* is farther from, or nearer to the *focus* (F) of the *lens*.

19. If the *radiant point* A. (fig. 25.) had been any where between the *lens* B C, and its *focus* F, the *pencil of rays* would have *diverged* after passing the *lens*, but in a less degree than before.

20. Let D E G, (fig. 26.) be any *object* placed beyond the *focus* F, of the *convex lens* B C; this *object* will send forth *innumerable rays* from every point of it, some of which falling on the *lens* B C, will form *innumerable pencils of rays*, each having the *lens* B C, for its base, and a certain point in the *object* D E G for its vertex. But by the nature of the *convex lens* each of these *pencils*, will be collected again into a point in the continuati-

then they consider another small portion of rays and find its focus, and then a third portion, &c. These *loci focorum* form a curve, which they call the *diacaustic curve*.

Vide *Simpson's Fluxions*, page 536. *Martin's Institutions*, vol. II. page 122. Or *Hays's Fluxions*, page 243. In which last, is a very full account of the nature of this curve; and the theorems for determining the *foci* of all kinds of *lenses* are elegantly investigated.

on of its *axis*. And if these *points of concurrence* are received on a sheet of paper, P. (on which no light is suffered to fall, but what comes through the lens B C) they will form a *picture* g e d, of the *original object* in an inverted situation. Thus the pencil of rays, D B C, which flows from the point D, will be collected again in the point d, (which is in the continuation of the axis D S, of that pencil. *Vide art. 10. and 13.*) Also the pencil of rays B E C, which flows from E, will be collected again somewhere in its axis E S continued; as in e; also the pencil B G C, will be collected in g.

21. If several objects are situated at different distances from a lens, and the nearest of them is many times the diameter of the lens distant from it, they may be esteemed as equally distant. For the *pencils* will, to sense, consist of *parallel rays* and will converge on a sheet of paper passing through the focus of the lens. (*Vide art. 14. and 15.*)

22. But if the objects are all *near the lens*, but at different distances from it: a sheet of paper which receives the picture of one object, will not receive the picture of another. Thus if the *sheet of paper* P. (fig. 26.) receives the *picture of the object* D E G, it will not receive the *picture of the object* H I, because *each pencil of rays* proceeding from *every point in the latter*, diverges much more than *those pencils* which proceed from *every point in the former*

former D E G, and therefore will not converge on the paper P, unless it is removed farther from the *lens*. In which case a *picture* of the object H I, will be formed on the paper P: and if we suppose the *object* H I, to approach towards the *focus* F, of the *lens*, and the *paper* P, to recede farther and farther from it; a succession of *similar pictures* of the *object* H I, will be formed on the *paper*, which will increase amazingly in magnitude as the *object* H I, draws nearer the *focus* of the *lens*. But when the said *object* arrives in the *focus*, no image can be formed; because the *pencils of rays* proceeding from the *object*, will, after their passage through the *lens*, become parallel, (*Vide* art. 17.) and consequently will never unite.

23. The truth of all this may be made evident by an easy experiment. Place a lighted candle at one end of a dark room, and taking any *convex lens* (as a glass of a common pair of spectacles) go to the other end of the room, and holding the *lens* between the candle and a sheet of paper, at such a distance from the paper as is nearly equal to the *focal distance* of the *lens*. A small *inverted figure* of the *flame* will appear on the paper. Approach the candle, still holding the paper in one hand, and the *lens* in the other; and very little variation will for some time be observed in the figure of the flame on the paper: but when you arrive nearer the candle,

candle, the *picture* will grow larger and larger, and the paper must be held still more and more remote from the lens. Approaching still nearer, the picture will exceed the bounds of the paper; and when the lens becomes but little more remote from the candle, than its *focal distance*, a very *enormous inverted figure* of the *flame* will be seen on the opposite side of the room, which will be larger according as the side of the room is more remote from the candle.

24. As it is the nature of the *convex lens* to collect a pencil of rays which falls on it either diverging, or parallel to each other: so it is the nature of a *concave lens* to disperse, or render parallel any pencil of rays, which falls on it in a *converging state*.

But it is not my business here to say more of the nature of lenses, than is sufficient to explain the manner by which the eye performs its function of making us sensible of the figures of enlightened objects, which sense we call vision; and for this purpose, it will be found that the chief furniture of the eye answers to the *convex lens* and the *sheet of paper* already described, and that the other parts display the most amazing contrivance to adjust the part which answers to the lens, in such a manner to that which answers to the paper, that the pencils of rays passing through

through the former, may always converge on the latter *.

25. The eye is nearly globular; it consists of three *coats*, and three *humours*. The external coat consists of two parts; one of which, D E F G (fig. 27.) called the *cornea*, is transparent, and more convex than the other, D H H G, called the *sclerotica*, which is opaque. The *cornea*, when dried, resembles a piece of transparent horn; from whence it derives its name.

26. Next within this coat, is that called the *choroides*; in the front of which, is a small circular hole, (P) called the *pupil*; this hole is surrounded with the *iris* *m n*, *m n* §. which is composed of two sets of muscular fibres; the one of a circular form, which contracts the *pupil*, when the light is too powerful for the eye; the other is composed of radial fibres, tending every way from the circumference of the *iris*, towards the center of the *pupil* P; which fibres, by their contraction, dilate and enlarge the pupil, when the light is weak, in order to admit more of its rays.

* This alone, where there are no proof, shews that there must have been an intelligent being, who was perfectly versed in optics, before an eye was formed.

§ The *iris*, is so called from its being of various colours; whence the eye is said to be black, blue, hazel, grey, &c.

27. On one side of the hinder part of the eye, the *optic nerve* L enters it, from the brain; and is finely expanded over the *choroides*, last mentioned. This expansion of the *optic nerve*, forms the third coat of the eye; and is named the *retina*. In its colour and contexture, the *retina* is not unlike extreme fine white linen. Its office is to receive the images of objects, in the same manner as the paper mentioned, art. 23.

28. At a small distance within the *pupil* P, is suspended the *chrystalline humour*, I, its form and office is the same as a *convex lens*, the inner surface of which is somewhat more convex than the other surface. It is composed of an extremely clear jelly-like substance, contained in a fine transparent membrane, called the *arachnoides*; from which proceed radial fibres, o, o, all around its edge; these fibres are called the *ligamentum ciliare*; and by joining to the circumference of the *iris*, m, m, not only keep the *chrystalline humour* (I) suspended, but also by their power of contracting and dilating occasionally, they alter the convexity of the *chrystalline humour*, and also shift it a little backward or forward. This *chrystalline humour* divides the eye into two cells, P and K K.

29. Lastly to keep it in its globular form, and also to preserve all the parts in a proper tension, and situation with respect to each other; the divine constructor of the eye, has filled

filled these two cells, P and K, with fluids ; which differ but little in *specific gravity*, and *refractive power*, from common water. That included between the *chrystalline humour* and the *cornea*, is called the *aqueous humour* (from its similitude to water) and that in the space (K, K) included between the said *chrystalline humour* and the *retina*, is called the *vitreous humour* ; this last is much the largest in quantity, but partakes of none of the qualities of glass, (which its name indicates) but transparency.

30. From this account of the construction of the eye, and what was before explained, it is evident, that if any pencil of rays (t u w fig. 27.) issue from any point, (as B) situated at a good distance from the eye, they will differ but little from parallel rays, (*Vide* art. 14.) and by falling on the middle of the cornea D E F G, they will go through the pupil (P) and converge to a point in the *axis* * of that pencil continued : (*Vide* art. 15.) which point will be nearer to, or farther from the *cornea*, according as the said *cornea* is more or less globose : in this *converging state*, they will strike on the

* The axis of any pencil of rays which enter the eye, is that particular ray which passes through the center of the pupil. Thus the ray f. (fig. 27) is the axis of the pencil r s t ; the ray u, is the axis of the pencil t u w, and y, is the axis of the pencil x y z.

chrystalline humour I. which will make them converge still more; and if the bottom of the eye is at a due distance from this *chrystalline humour*, the said pencil of rays (*t u w*) will converge on the *retina*; and this is always the case in a well formed eye. Thus the pencils of rays *q r s*, *t u w*, *x y z*, which flow from every point of an object *A B C*, are, by the humours of the eye, made to converge on the *retina*. and there form an inverted image of that object, in the same manner as the image was formed on the paper in the experiment, art. 23. The converging power of a well formed eye being such as will unite any pencil of rays to a focus on the *retina*, which enters the *pupil* *P*, in a direction nearly parallel.

31. It will not be difficult from hence, to account for the several impediments which may hinder a pencil of rays falling on the cornea, from uniting on the *retina*: for first, if the eye is perfectly formed, and the object be held very near it; then the several pencils proceeding from every point in that object; will pass through the cornea and enter the pupil, in a state too diverging to be united on the *retina*. To remedy this, a *convex lens* applied between the eye and the object will render those pencils nearly parallel, before they enter the eye, (*Vide* art. 17.) and then the refractive power of the eye will unite them

them on the retina. It is on this account the single microscope becomes necessary.

32. If the cornea (a b c fig. 28.) is formed too globose, or the *chrystalline humour* (I) is too convex; any pencil of rays coming from a point (B) placed at a good distance from the eye; will converge to a point (P) short of the retina; because the *converging* power of the eye is too great for distant objects: and this is the case with short sighted people, who cannot see objects at a due distance; but are obliged to make the pencils proceeding from those objects enter the pupil more diverging, either by holding them close to their eye; or by applying a concave lens between the eye and the object; by either of which methods the said pencils become more diverging before they enter the eye, and by the converging power of such an eye, they will unite on the retina, and form a picture; which produces the sensation of vision.

33. On the contrary when the *cornea*, (a b c, fig. 29.) or the *chrystalline humour* I, is less convex than it should be, (arising, perhaps, from a deficiency of juices to keep the eye to its proper globosity.) The converging power of the eye, will be too little to make a pencil of rays proceeding from any point of an object, unite on the *retina*; but each pencil will unite in a point beyond it. Thus the rays proceeding from the point B, unite at the point P, beyond the *retina*; to remedy

remedy this inconvenience, a *convex lens* must be applied before the eye; which will cause the said pencil to enter the pupil P. in a *converging state*, and by receiving an additional converging power from the humours of the eye; the said pencil is made to converge on the *retina*.

34. The eye is naturally endowed with a power of varying its *converging power*, by means of the *ligamentum ciliare*. (*Vide art. 28.*) Thus when the objects are near, so that in a good eye the pencils would unite a small matter beyond the retina; we can shift the *chrystalline humour* forward, and by that means throw the converging point on the retina: on the contrary, when very distant objects are viewed, the *chrystalline humour* is shifted a little backward.

35. By what means the image thus painted on the retina, gives us the sensation of vision, is beyond the limits of human conception: but this we know, that when any of the causes beforementioned, forms the images either beyond, (as in fig. 29.) or short of the retina. (as in fig. 28.) The sense of vision is imperfect and obscure: and that those imperfections vanish, when such remedies are applied, as may make the several pencils of rays, proceeding from an object, united on the retina, and thereon form its image.

C H A P.

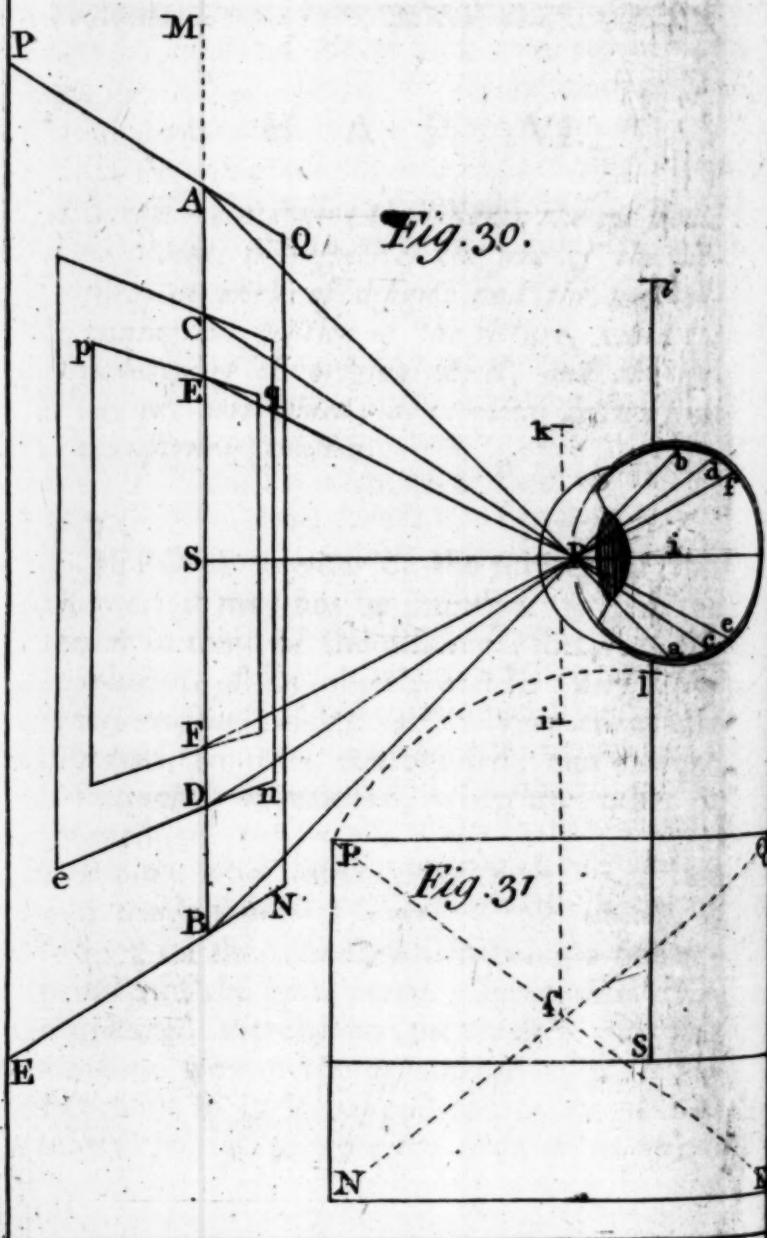


C H A P. VI.

A digression concerning the distance the eye should be placed, in respect of the size of the picture, in works of design: and the most advantageous position of the picture, when the situation of the original object, and the eye, are pre-determined; as in views drawn from an assigned situation.

BEFORE I enter on the subjects of this chapter, it may not be improper to put the reader in mind of the difference between the *appearance* of an object, and its *perspective representation*; which altho' very essentially different, are often confounded; and the one is frequently mentioned, when the other is meant.

From what was said, page 75, we find the picture of every visible object, is formed on the *retina*; which being a fine expansion of the optic nerve, a sensation is communicated thereby to the brain; and this sensation we call the *appearance* of that object: but as it is beyond the extent of human sagacity, to trace the manner in which
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any impression made upon a nerve, operates on the sensorium; it must suffice our purpose to esteem the picture of any object which is formed on the *retina* by the humours of the eye, as the *appearance* of that *object*. And this we may the more readily do, because experience teaches us, that as this picture varies in shape, strength and magnitude; our ideas of the figure, distance and size of the said object which causes it, vary in a corresponding manner.

It is for this reason, that when we want to inform ourselves of the true shape of any object, or of any part of it, we endeavour to place that object, or part, and the retina, parallel to each other; because by that means the picture on the retina, will be similar to the object we contemplate: (*Vide* page 75. and § 71.) and according as this can be, more or less, exactly done, our idea of the true figure of the object is more or less perfect*.

The *appearance*, then, of an object, is formed on the retina: but the *perspective representation* of an object is formed on the plane of the picture: (*Vide* page 2.) and altho' the same object will always have the same *appearance*,

* A line *Si* (fig. 30.) passing through the center of the eye (*i*) and the center of the pupil, (*P*) is called the *axis of the eye*. The *axis of the eye* is therefore perpendicular to the bottom of the *retina*; and when it cuts any plane at right angles, that plane is parallel to the bottom of the *retina*. (E. 11. 14.)

when

when viewed from the same place; yet it may have ten thousand different *perspective representations*; because the rays proceeding from the said object to the eye, (and which cause the *appearance*) may be cut by a plane in an infinite number of situations and positions: and these various sections of those rays, will form so many different *perspective representations*, all which will afford the eye the same sensations, (that is, all these will have the same *appearance*) as the real object itself: and tho' some of these *representations* may be totally dissimilar from the object they represent; and may seem void of shape and uniformity, when cursorily viewed; yet any such will appear natural and uniform, to an eye which views it from its proper point; provided the situation of the original object and the eye were so chosen, that the latter might view the former to advantage, supposing the *plane of the picture* was out of the question.

But if the situation of the eye and the original object are such, that a part of it only can be seen; or if by reason of its distance or obliquity, the shape and distance of the parts, are rendered so obscure and imperfect, as to afford the eye neither pleasure nor information; in such a case, the *perspective representation*, (whose business is to produce the same effect as the original object, (*Vide page 2.*) must labour under the same disadvantages
in

in whatever situation we suppose the plane of the picture to be placed, between the eye and the object.

It is not therefore the situation of the eye with regard to the picture, that can produce a bad effect; for altho' by placing the *plane of the picture*, very obliquely between the eye and the object, the *representation* may be totally unlike the *appearance* of that object, yet all this dissimilarity vanishes, when the eye is placed to view the picture, in the same position in which it was supposed, when the picture was drawn. Thus the rectangle A B C D, (fig. 15.) is so placed, that it may be viewed to advantage by the eye at I; but its *representation* a b c d, on the inclined picture E N V L, is an *irregular Trapezium*: yet the eye, at I, sees nothing of this irregularity, but the mind has the same sensations as are produced by the original rectangle itself, or its similar *representation* on the upright picture P Q E N.

Nevertheless, it must not be understood, that all situations for the picture, with respect to the eye and the original object, are equally proper; for altho' the most *distorted perspective representations* are equally as just and true as those which are more similar to the original object; yet in pictures which are for general inspection, we must chuse such a position for the eye, as we may imagine any person who views the picture, will naturally

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place

place his eye in : for then the picture will always be viewed to advantage, and will convey at first sight, the idea which the artist intended.

Suppose P Q E N (fig. 30.) is a paper circumscribed with a border, within whose limits a perspective representation is to be made: our enquiry then is, (esteeming this drawing as an object for inspection) in what position, and at what distance, any indifferent person will place his eye, in order to view this drawing when it is finished.

From what was observed, page 79, it is evident, that the picture E N P Q, will be viewed to most advantage when the *axis* (P S) of the eye is perpendicular to it; because then the said picture will be parallel to the bottom of the retina: also if the said axis, P S, cuts the picture in the middle, (S) any equi-distant parts of its extremities, (as A and B) will be equally oblique to the eye: that is, the rays A P, and B P, proceeding from those points, will enter the pupil (P) with the same Obliquity: and the *appearance* (a b,) of these points will be the nearest to the middle of the retina possible; and consequently the whole of the picture will be viewed to the most advantage when the seat of the eye * (S) is in the middle

* The point (S) where the picture is cut by a perpendicular P S, drawn to it from the eye, is called *the seat of the eye*: and sometimes this point (S) is called *the center of the picture*; and the said perpendicular P S, is called *the distance of the picture*.

of it: it is for this reason, that when people want to see the whole of any *portable object* at one view, they always direct the axis of their eye, so as to strike the middle of that object in a perpendicular direction. They also hold it at such a distance, that they may see it with ease and distinction: if they hold it too near, they cannot see the whole; and if they place their eye too far off, they cannot distinguish the minute parts of it.

If the *distance* (P S, fig. 30.) of the eye from its seat (S) on the picture, is equal to the distance which any point (A) on the picture, is from the said seat of the eye: then the ray, PA, proceeding from that point to the center of the pupil, (P) * will cut the axis of the eye P S in an angle (A P S) of 45° (E1. 32. Cor. 4.) and this is the greatest obliquity at which any point can be visible: for if any point (as M) is more oblique, the pencil proceeding from that point, will not be united on the retina by the humours of the eye: (*Vide the note, page 67*). Wherefore, if the seat of the eye (fig. 31.) is any where pre-determined, the shortest distance for the eye that can be allowed, is (S I equal to P S,) the distance which the

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* Of the *pencil of rays*, proceeding from the point A, only the ray A P, need be regarded, which is the *axis* of the said pencil (*Vide note, page 74*.) for the point (a) where the said axis A P, cuts the *retina*, will be the point where that pencil unites and forms the *appearance* (a) of the point A, (*Vide art. 20 and 30, of the last chapter.*)

most distant part (P) of the picture is from its center (S.) (*Vide the note, page 82.*)

But altho' the point A, (fig. 30.) is visible, yet it cannot be seen distinctly: and the point C, or D, whose distance (C S or D S) from the seat of the eye (S) is only two thirds of the distance (P S) of the eye from the picture, will be a much better boundary for its limits: because the rays C P, or D P, cut the axis of the eye less obliquely, and form the *appearance* (c and d) of those points, nearer to the center of the retina. But if the distance of the farthest limits of the picture from the seat of the eye (S) does not exceed E S, or F S, which is only half the distance (P S) of the eye: then the whole *appearance* of such a picture, will occupy a space on the *retina*, whose diameter will not exceed e f; which space will differ but little from a plane surface, and being parallel to the picture, (p q e n,) the figures on the picture and the retina, will be very nearly similar.

Instead then of making the distance of the eye (S I, fig. 31) equal to (P S,) the distance of the most remote extremity of the picture from its center, it will be more proper to make the said distance of the eye, equal to once and an half that distance (P S,) or if the distance (S i) of the eye from the picture, is taken equal to twice (P S), it will be still nearer the distance which a person will naturally place his eye at, to view the picture when
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it is finished; because at this distance he will easily see the whole, without losing sight of the minute parts; which, as was observed before, is what every spectator naturally aims to do.

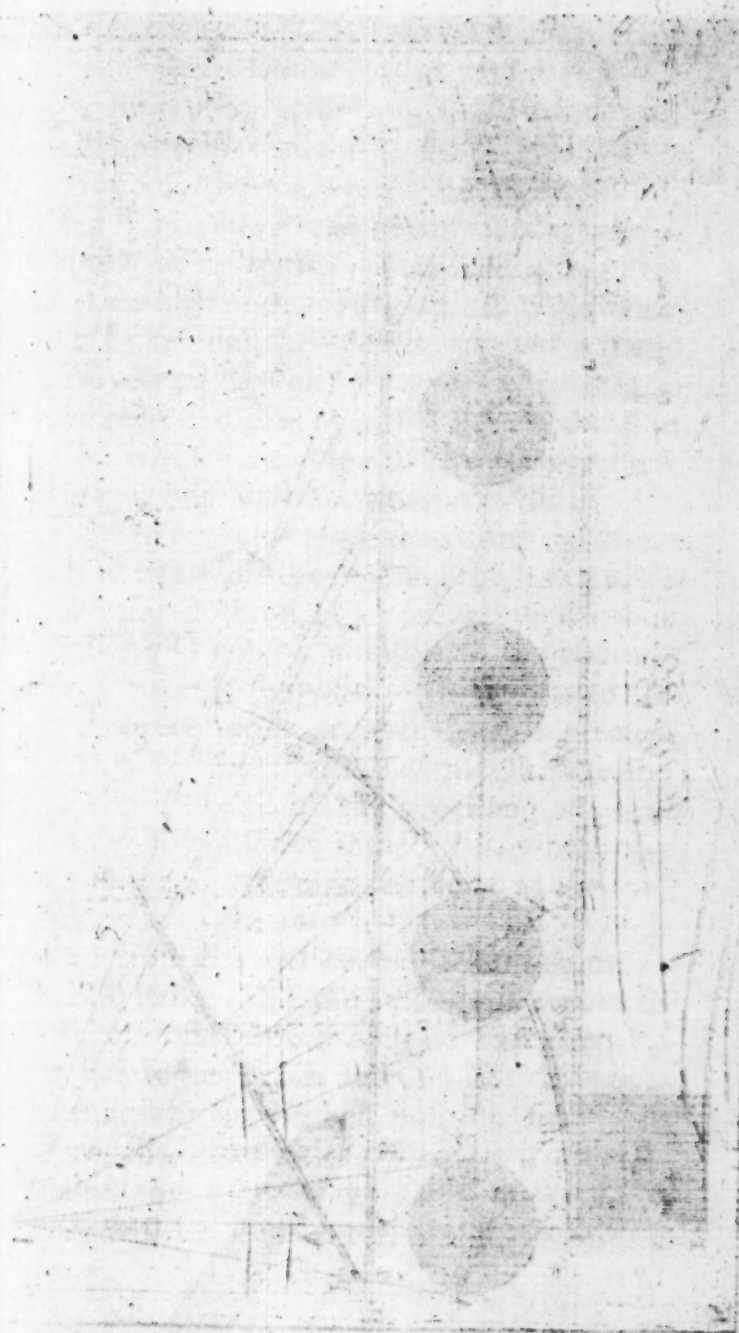
If the seat of the eye (S) is assumed in the middle of the space ($PQEN$, fig. 31,) allotted for the picture, all the extremities of that space will be the nearest to it possible; and the distance of the eye (Si , or Sk) when taken equal to once or twice (SP , SQ , SN , or SE), the distance of the corners from the seat of the eye (S), will be shorter than it can possibly be, when the said seat of the eye is not in the middle of the picture ($PQEN$). And this, together with what was said before, seems to argue, that the seat of the eye ought always to be in the middle of the picture: but whether this sometimes might not destroy the *beauty of composition*, I leave to the superior taste of the artist; confessing myself a very incompetent judge of that disposition of the parts, which constitutes a fine picture.

I cannot quit this subject without putting the reader in mind that if he works with a short distance, his piece will appear vastly distorted to an eye that views it at a greater distance than the proper one: but if he uses a long distance (or one not less than twice the distance of the farthest extremity of the picture from the seat of the eye) no deformity will arise by viewing it closer than the proper distance,

distance, consequently it is most proper to work with a long distance; altho' attended with a little more trouble, by reason of the vanishing points being sometimes so distant from the center of the picture, as to exceed by much the bounds of the paper we are drawing on: it is for this reason I have been obliged to use shorter distances in some of my examples, than are recommended in the preceding observations: but we shall find a remedy for this inconvenience in the course of the seventh chapter.

In the foregoing considerations, the size of the picture was supposed predetermined; and it was proposed to determine such a distance and position of the eye, that any thing designed on this picture by the rules of perspective, might, to a common observer, appear natural and void of deformity. The chief use of them therefore relates to works of design: but in drawing *real objects* from nature, we must proceed in another manner; because in this case the *position of the eye, and the original object to each other*, are predetermined; and it remains to determine the position of the picture *between* them, so that the *representation* may *appear* as natural as possible to a common observer, viewing it at random.

The solution of this problem will not be difficult, if we reflect that when a person looks at any object with attention, he turns his eye towards it in such a manner, as to see each



each extremity of it at once, with the same ease. (If on account of his vicinity to the object, he cannot see both extremities at once, he will go farther off till he can). To illustrate this, suppose the five circles and the point I, fig. 32.) to denote the situation of the eye, and a row of columns to each other; it is evident, that although we suppose the eye fixed at I, yet by turning the head about, the spectator will see any particular column directly; thus, if he directs the axis of his eye along the line I m, he will see the column G, to the greatest advantage, but will not be able to distinguish the column H at all: on the contrary, if he directs the axis of his eye along the line n I, he will see the column H in a direct view; but the column G will be undiscernible: but according to our supposition, he wants to see both these columns at once to the greatest advantage possible: and for this purpose, he will direct the axis of the eye along the line I X, so as to *bisect* the angle A I h, which the whole range of columns subtends at the eye. In this position the rays h I, and I A (which proceed from the extremities of the columns H and G) strike (IX) the axis of the eye, with the same obliquity; which is measured by the equal angles h I X and A I X. It is evident, that this is the best position for viewing *both* the extremities; because if the eye is turned any other way, although one extremity will be seen better, the other will be

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seen more oblique and obscurely.—Having thus determined the position of the eye, the position of the *picture* cannot be mistook; as it was before observed, that if the picture *p p* was placed parallel to the retina, or perpendicular to the axis of the eye (*I X*), the figures on it and the *retina*, will be nearly similar; and all objects will be represented on the picture, nearly as they *appear* on the *retina*; and we shall not err much when we call such a representation the *appearance* of those objects. The distance of the picture *p p* from the eye is immaterial; because a variation of the distance, will only vary the magnitude of the objects on the picture, but will not vary their shape. Hence it follows, that since the *representations* of the columns on the picture *p p*, is nearly similar to their *appearance* on the retina; the representation of these columns, on the picture *P P*, placed parallel to them, is very unlike their *appearance*: and although both these pictures *P P* and *p p*, will have the same effect when viewed from the point *I*; yet from any other point, the picture *P P* will look very disagreeable and deformed; and a common observer will say that it is not a just *representation* of a row of equi-distant and equal columns; the contrary will happen to the picture *p p*, which will always be viewed in the position the eye was supposed in when it was drawn. And the objects drawn on it having shapes similar
to

to what every man has seen, he will pronounce the piece well drawn, and agreeable to nature.

I have, perhaps, been too prolix in these digressions ; but the first appeared to me necessary to explain the second : the subject of which being partially or erroneously understood by some writers, has occasioned mistakes and controversy, which rendered a full discussion necessary.

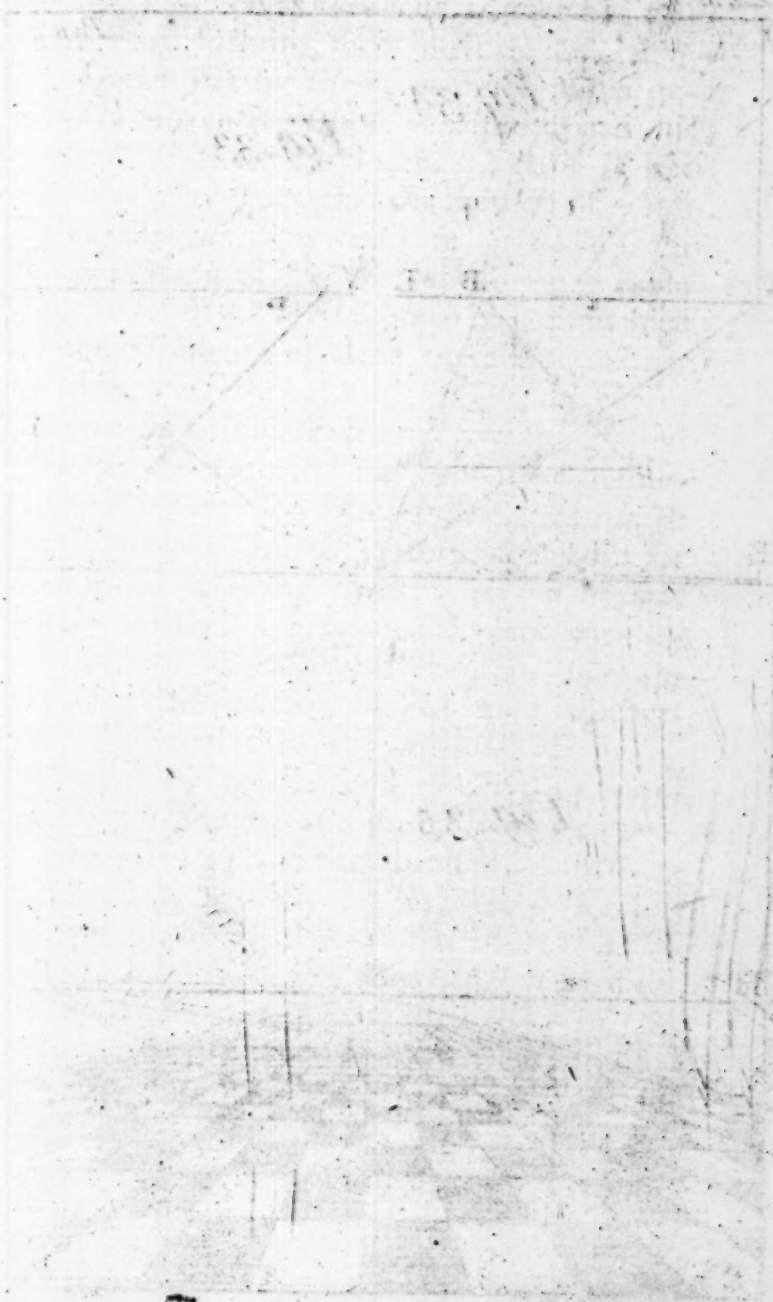
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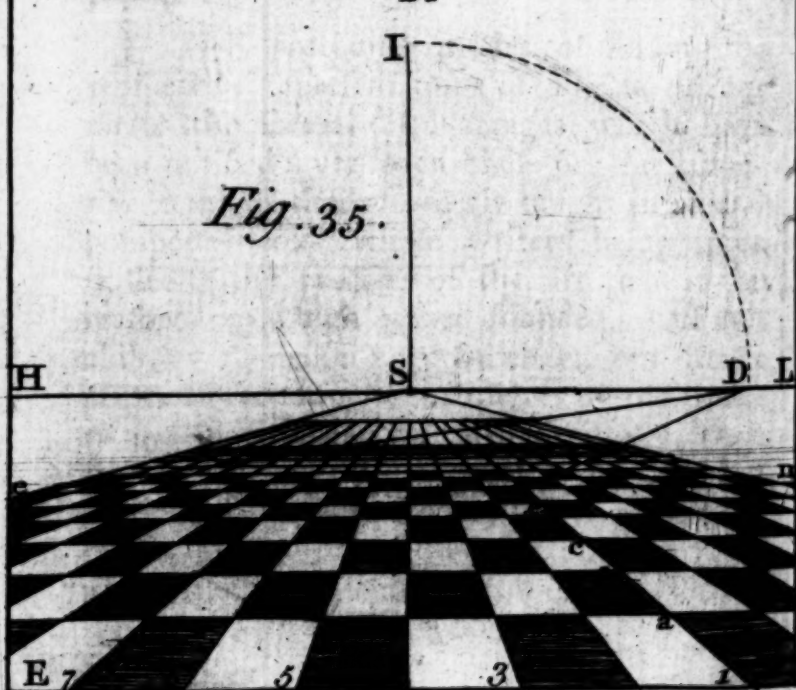
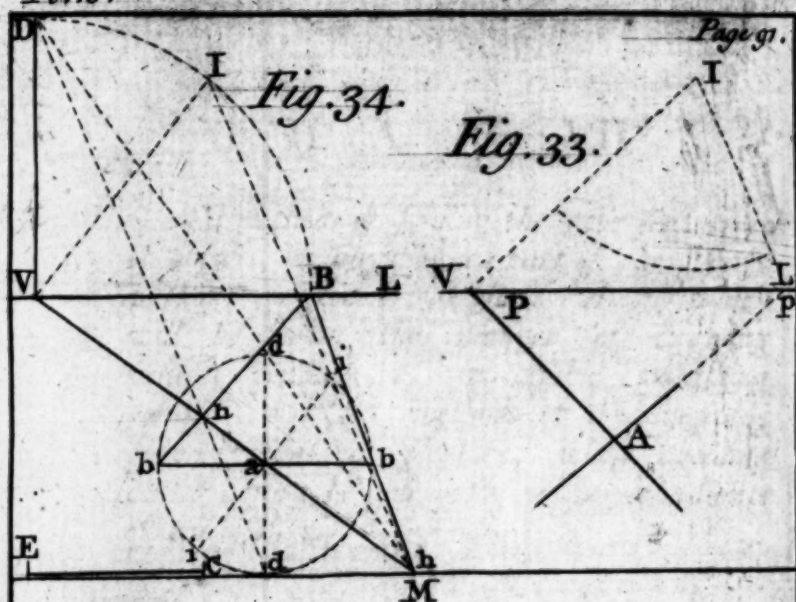


C H A P. VII.

Universal methods of finding the representations of objects, in any assigned part of the picture without a geometrical plan. Among which will be shewn the manner of drawing visuals tending to inaccessible vanishing points; and the methods of dividing any given visual at pleasure, whose dividing center cannot be laid down within the limits of the picture.

THE preceding modes of finding the perspective representations of objects, labour under the several disadvantages which have been notified: yet even these are far superior to the methods formerly taught in several pompous books, whose writers being more skilled in the practice of the art than in the reasons on which it are founded, had not abilities to make improvements, but strove rather to recommend themselves by the beauty and magnificence of their examples, than the convenience of their rules. At the head of these is Andrea Pozzo, whose work, in two volumes folio, printed at Rome, in 1717 is the most ornamental book on this subject





I have ever seen : but his two methods of working are such as make us admire his patience in performing such intricate and laborious examples by them, as much as his genius in inventing those examples and his neatness in their execution. Yet it is but just to acknowledge that the method of using the auxiliary *entering line* § 74, is taken from the beginning of his second volume. Bosse and the Jesuit are little more to be commended for the excellence of their methods : neither of them ever thought of finding the representation of an object without having a ground plan first determined, and even some mathematical writers on this subject, although they have extended the theory of this art far enough for drawing objects, on an *upright picture*, without a geometrical plan, yet they do not avail themselves thereof in their applications to practice, indeed their discoveries only amount to the methods of finding the *vanishing points*, and *dividing centers* of such original lines, as are parallel to the horizon, to which plane they limited all their attention. It was left for Dr. Brook Taylor, first to divest the theory of this art from all ideas of locality, from whence he derived the most concise and universal *methods of perspective* of which we shall treat in this and the succeeding chapters.

We may geometrically draw a right line figure of any proposed shape and dimensions,

sions, if we can draw a line which shall cut a given line in any proposed point, and form with it a given angle : and if at any proposed point we know how to put a line equal to one of an assigned length : in like manner we may *perspectively* draw the representation of any right-lined figure, if we can draw two lines cutting each other in such a manner as to represent two original lines forming any proposed angle ; and also that they may be the representations of lines of a proposed length. How this is to be done, when the original lines to be represented are in a plane parallel to the picture has been shewn in problem 1 and 2, § 65 and 68. It is now to be shewn how the same may be effected, when the lines to be represented, are situated in a plane oblique to the picture ; but whose *vanishing line*, with the place of the eye (*Vide* page 24.) is always given.

P R O B L E M III.

LXXV. Through any proposed point (A, fig. 33) in a given *visual line* (A P) whose *vanishing point* P, with the *place of the eye* I, is given, to draw another *visual line* (A p) so that these two *visual lines* (A P and A p) may represent lines forming a given angle.

1st. Draw the line, I P, joining the place of the eye (I) and the vanishing point (P) of the given *visual line* (A P)

2d. Draw

2d. Draw $I p$, making with the line $I P$ the angle $P I p$ equal to the given angle which the *original lines* are supposed to form, and cutting the vanishing line ($V L$) in p .

Lastly, Draw $A p$ which will be the visual sought, the reason of which is evident from § 25 and § 19.

P R O B L E M. IV.

LXXVI. The *visual line* ($V M$ fig. 34) of any *original line* situated in an *original plane*, whose *vanishing line* $V L$ together with the *place of the eye* (I) is given: from any assigned point (a) in that *visual*, to cut off a part ($a h$) equal in *representation* to a given length ($E C$.)

G E N E R A L S O L U T I O N.

1st. At the given point (a) put the *representation* ($a b$) of a line equal to $E C$ and parallel to the *vanishing line* ($V L$) by prob. 1. § 65.

2d. Through the *vanishing point* (V) of the given *visual*, ($V M$) and the given point, (a) draw any two lines parallel to each other; as $V D$ and $a d$, $V I$ and $a i$, or $V B$ and $a b$, &c. and make that drawn through the *vanishing point* (V) equal to ($V I$) the *distance* of the said *vanishing point* (*Vide* § 9.) and make its parallel drawn through a , equal to $a b$.

Lastly

Lastly, Draw a line, as D d, or I i, or B b; joining the extremities of these parallel lines, and cutting the given visual in h, then is a h the *representation* of a line equal to E C. Q. E. I.

DEMONSTRATION.

This is in effect the same method as was taught in § 48, which the reader is desired to read carefully over. Suppose then that a c (fig. 12.) the *representation* of A C, is found by the means there recommended; then I say that if thro' c, (which is the *representation* of C) a line c f, is drawn parallel to the *distance* S i, or the *scale line* m g, (*vide* § 47) and terminated by the line g i, (which determines the *representation* a, of the point A) this line c f is equal to c e, (which is the *representation* of a line at c, parallel to the entering line, and equal to A C).

For, because S i is parallel to m g, the triangles m c h and c S i are similar, wherefore

$Sc : cm :: ic : ch$ by E. 6. 4. and

$Sc : (Sc + cm =) Sm :: ic : (ic + ch =) ih$
by E. 5. 18

But since the triangles S d c and S m n are similar, we have $Sc : Sm$ (or $ic : ih$) :: cd ($= ce$) : nm ($= CA$)

that is,

$Sc : Sm$ (or $ic : ih$) :: $ce : CA$;
but because of the similar triangles i c f and

ihg, we have $ic:ih::cf:hg(=CA)$ consequently $ce:CA::cf:CA$, (by E 5.11.) wherefore cf is equal to ce by (E 5 9) from whence we may gather that if cf is drawn parallel to Si , and made equal to ce the line joining f and i , is the same as the line joining g and i ; and therefore must cut the visual line in S in the same point a .

COROLLARY. I.

LXXVII. There other methods of solving this problem as will appear by observing fig. 12. where, because of the similar triangle, ac and Si , we have $ac:as::cf$ (or ce): si : wherefore to solve our problem we have nothing to do but to cut cS into two parts; ca and aS , which shall be to each other as ce to si , and as there are several ways to do this (*Vide* E. 6. 10) there are as many ways to solve the problem, either of which may be used as pleasure or convenience may dictate, as will be shewn occasionally.

COROLLARY. II.

LXXVIII. If the distance VI (fig. 34) of the *vanishing point* of the given *visual* VM . is laid on the *vanishing line* (VL) from V to B , then the line joining B and b cuts the given visual in h , and solves the problem, in this case we have no occasion to transfer a to any other line.

COROLLARY

COROLLARY III.

LXXIX. If a i (fig. 34.) is drawn parallel to the distance IV , and made equal to a b , then Ii cuts the given visual VM in h , and solves the problem. But in this case a i makes the same angle with a b , as VI makes with the *vanishing* line VL ; which is equal to the angle which the original line, whose visual is VM , makes with the entering line (by §. 20.) and since this angle is commonly known, we may, by the help of this consideration, divide any *visual line* at pleasure, by lines drawn to the place of the eye, (I) which is always given.

Before I proceed any farther, I shall give some examples of the application of these four problems.

EXAMPLE I.

To construct a floor of squares whereof the sides are parallel, and perpendicular to the picture. *Vide* fig. 35.

EN is the entering line, which is equally divided in any number of points, 1. 2. 3. 4. 5. 6. 7, from which lines are drawn to the center (S) of the *horizontal* line HL , which lines will represent the perpendicular sides of the squares (by §. 21.)

The distance (IS) is laid on the *horizontal* line from S to D , then (by prob. 4. § 78.) the line IS is cut in the points a , b , c , d , e , and f ,
by

Fig. 36

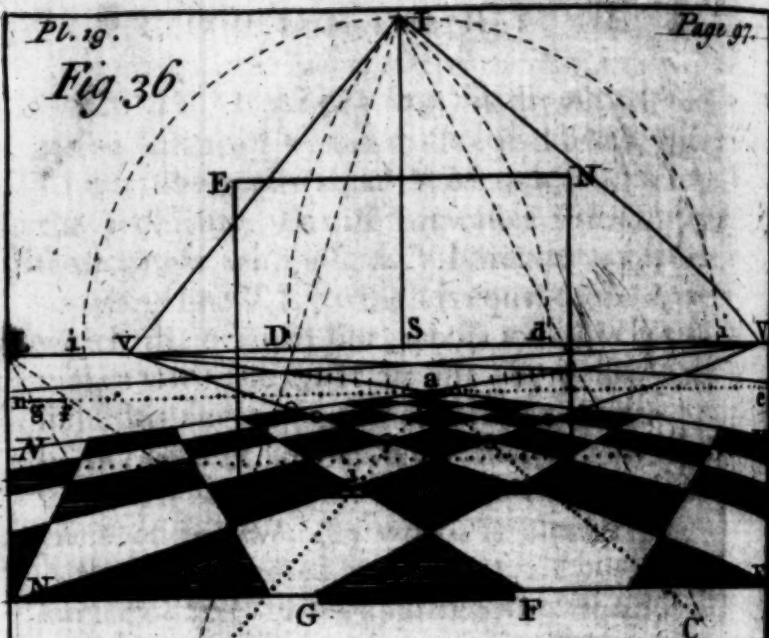
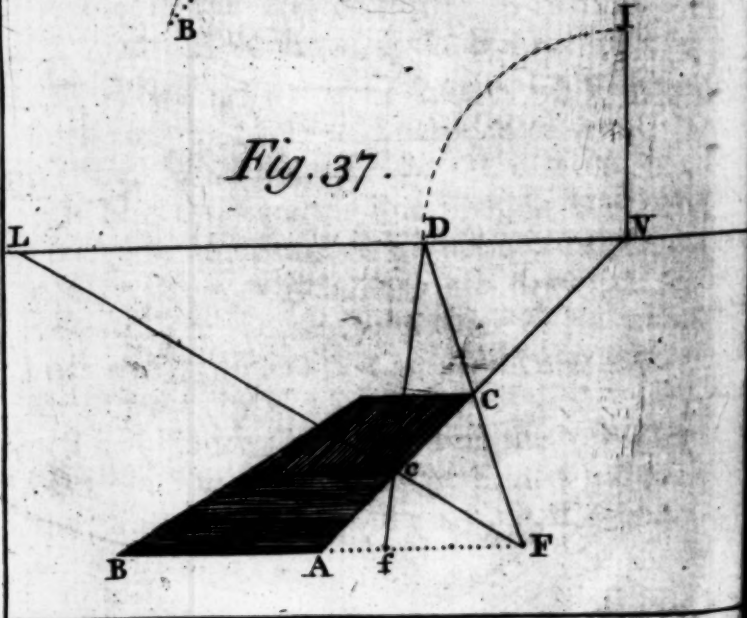


Fig. 37.



by lines joining D, and the several points 1, 2, 3, &c. Lines then are drawn through the points a b, c, d, &c. parallel to the *entering line*, and these will represent those sides of the squares which are parallel to the picture, and since there was six divisions laid down on the *entering line*; we shall by this means get the representation of thirty-six squares—if we would continue them so as to fill up the width of the picture: on the line e n, which passes through (f) the corner of the most remote square, lay the dimensions which the side of a square has at that point, as often as it can be contained within the limits of the picture. Through these divisions, on e n, and the point S, draw lines, and you will have six rows of squares, extending the whole width of the picture. If you would have more rows, through the 1st division, next n, draw a line to S, then lay a ruler by D, and the several divisions on e n, cutting the line n S in as many points as there are divisions on e n, through which points, lines drawn parallel to the *entering line*, will give so many rows of squares; which may be continued to the sides of the picture, in the same manner as done before. The number of rows may, by repeating the operation, be increased to infinity, but they would grow so very small as to be undistinguishable; thereby conforming to nature, which perspective never fails to do.

H

It

It will be more convenient in common practice, after having drawn a sufficient number of lines to S, to cross all these lines, by a line drawn from any of the divisions in the *entering line*, to (D) the distance laid down on the *horizontal line*, as the line 7 D, then through the several points in which this line, 7 D, cuts the lines tending to S, draw lines parallel to the *entering line*, and the work is finished. The reason of all which is evident from § 78.

E X A M P L E II.

To construct a floor of squares whose sides are oblique to the picture, fig. 36.

Through I, draw I v, making the angle, I v S, equal to that which you intend one of the sides of each square shall make with the *entering line*, EN. (*Vide* § 20. and 24.)

2d. Through I, draw I V, making the angle v I V, equal to 90° . (*Vide* § 25.) because a square is always right-angled (E 1. Def. 30.)

3d. Assume any point A at pleasure between the *entering* and *vanishing lines* (E N and V L) and through this point draw lines to the *vanishing points* V and v, and produce them till they cut the *entering line* (as in this case they do) in E and N. Also through A draw EN, parallel to the *entering line*.

4th,

4th. Assume on the entering line, $E N$, any distance, as $F G$, for the side of each square, and (by Problem 1, § 65) cut off from the point A , a part of $E N$, equal to it, and lay this Distance $A 1$, repeatedly on the line $E N$, each way from the point A , as often as you have room on your paper, as shewn by the large dots on the line $E A N$.

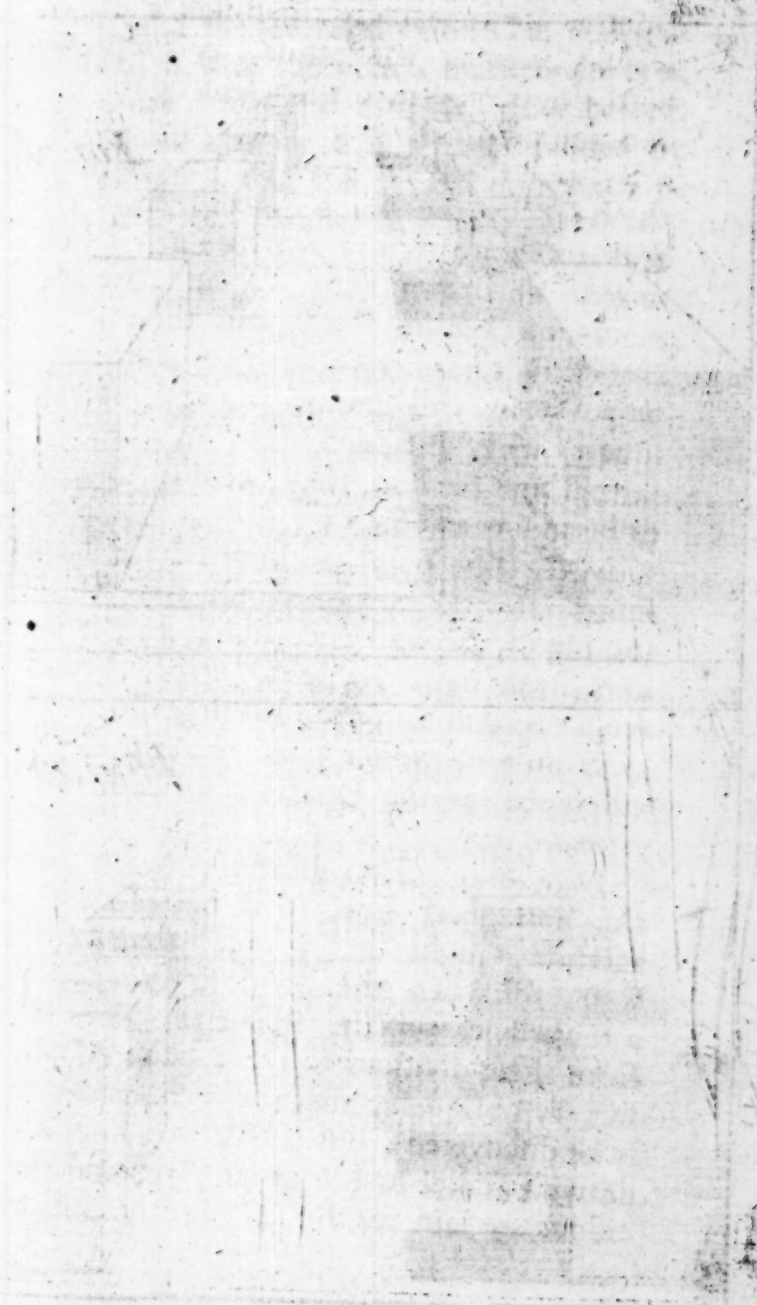
5th. By § 78, lay the *distance* $V I$, on the vanishing line to D , then a ruler laid by the point D ; and the several dots on $E N$ gives the points marked o on the line $V N$. Also if the distance $v I$ is laid on the vanishing line to d : then a ruler laid by d , and the same dots on $E N$, gives the points marked o on the line $v E$: lastly, thro' the several points thus found on $V N$, draw lines to v . Also thro' those found on $v E$, draw lines to V : then will these lines crossing each other, form the representation of several squares more or less in number, according as $F G$ (which was assumed for the side of each square) is smaller or larger.

The number of squares thus found may be continued *ad libitum*, if thro' a , (which is the most remote corner of the farthest square already found) a line $e n$ is drawn parallel to the *entering line* ($E N$): by the help of this line let the whole work be repeated in the same manner as was done before, by means of $E N$; that is, 1st by problem 1, § 65, cut off from $e n$, a part ($f g$) equal to $F G$. 2d.

from a, lay repeatedly the distance f g each way as often as you please, as is represented by the dots on e n, a ruler laid by these dots, and the point D, gives the dotted points on a V; and those on a v are found by laying the ruler by the point d, and each of the dots on e n from a towards n. Lines are now to be drawn thro' V, and the several points marked on v a; and these must be crossed by lines drawn thro' v and the dots on a V: by this means a great increase of squares will be added: but as they recede farther and farther, they become so small as to be indiscernible.

N. B. The point a, might have been assumed at first, which would have saved the trouble of a second operation; but it would have been more liable to error, because a small one committed in placing the distances exact on e n, would have made a very great difference, as the squares advance in the *fore ground*.

Note also, That if the point a is near the horizontal line, it is better to use the method taught in § 79, than that of § 78, for when the divisions are laid from a towards e, on the line e n, lines drawn from these divisions to D, cut the visual a V, so very obliquely, that the sections can hardly be discovered; but if thro' a, a line, a C, is drawn parallel to I V (*Vide* § 79) and the divisions are laid on this line (as the dots exhibit)



Pl. 10.

L page 101.

Fig. 38.

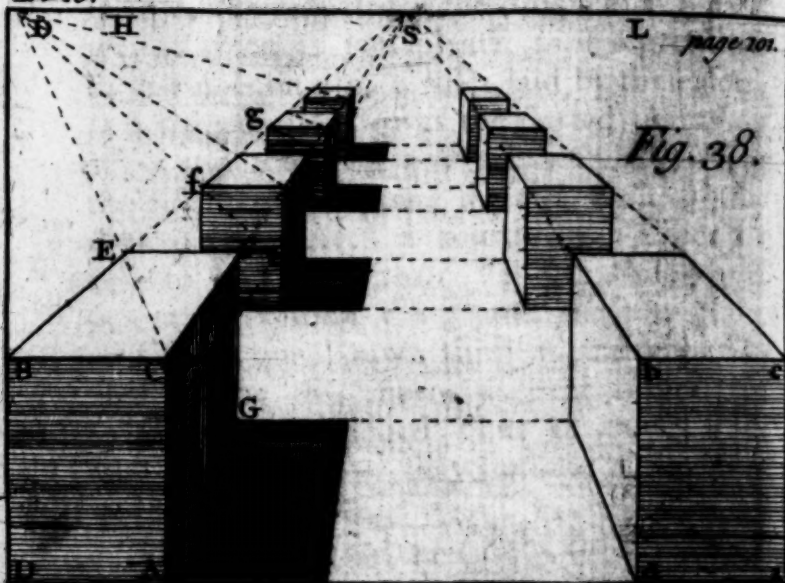
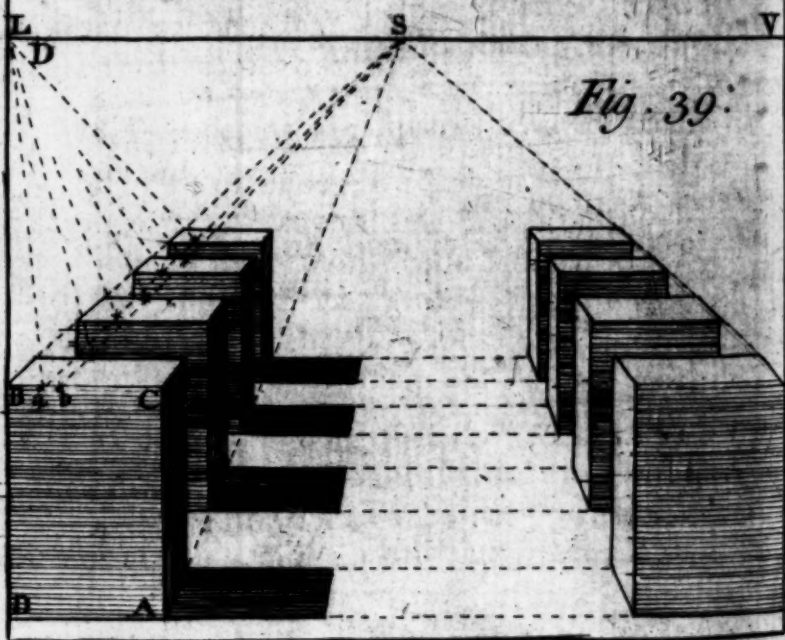


Fig. 39.



bit) instead of $e n$, then a ruler laid by the point I , and the several dots on $C a$, will cut the visual a V in the points marked thereon, much more direct and distinct, than before. In the same manner a v will be divided by lines joining I and the several dots on a B , which is drawn parallel to $v h$.

R E M A R K.

Most writers on this subject seem to regard that kind of obliquity only, which makes with the picture an angle of 45° . They assume the center (*S Vide fig. 36*) and the distance $S I$, of the *Horizontal line*; this distance they lay each way on the *Horizontal line* from S to i and i ; and whenever they speak of a rectangle whose sides are oblique to the picture, they generally make one tend to one point of distance (i) and the other to the other, in which case the sides of the figure are always inclined to the picture in an angle of 45° . for the angle $I i S$ is equal to 45° . (by *E 1. 32. Cor. 4*) and this is equal to the angle which any original line, whose visual tends to i , makes with the *entering line* by § 20. But the method here taught regards no particular obliquity, but all are managed with the same facility. Also this method of continuing the squares *ad infinitum*, is general in all cases.

H 5

R E M A R K.

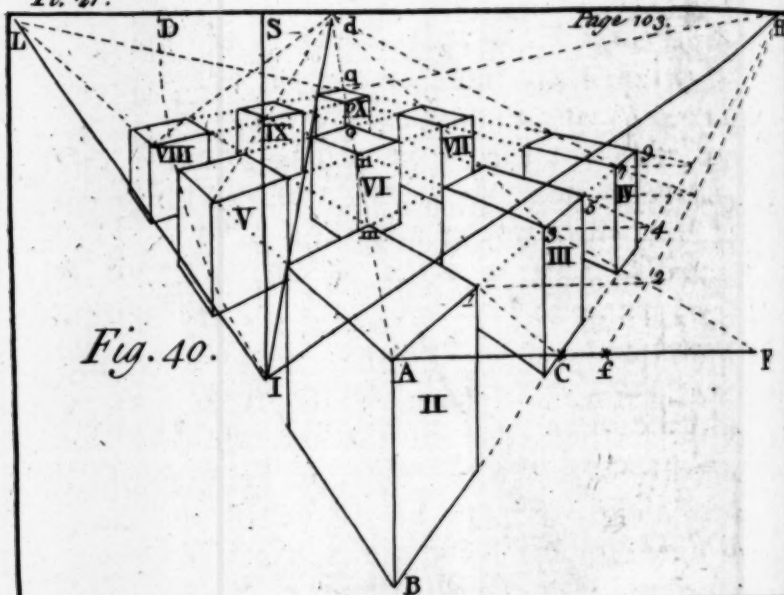
R E M A R K.

It is to be observed, that as the squares advance in the *fore ground*, and to the extremities of the sides of the picture, they become very distorted and by no means convey to the mind the idea of squares; because they cannot be viewed from the proper distance S I, as they subtend an angle at that distance, greater than the eye can observe (*Vide* page 83) but if a space be included about the center (S) by a square frame P Q E N of a due proportion to the distance S I as mentioned in page 84, all the deformities will be excluded, for the squares included in the frame P Q E N, appear regular, altho' viewed at a much greater *distance* than the proper one.

E X A M P L E III.

Fig. 38 represents two rows of *upright prisms* standing on square bases, having one side parallel to the picture, for which reason the other sides must tend to S the center of the picture. (By § 21) A B C D, the parallel front of the first, is drawn according to any assumed or given similarity; and from it all the other parts are determined. Thus S D, the *distance* of the *Horizontal line* is laid from S to D, and the visuals B S, C S and A S are drawn: then C D is drawn cutting the visual B S in E (*Vide* § 78) thro' E, E F is drawn parallel

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parallel to B C, cutting C S in F. Lastly F G drawn parallel to A C compleats the first prism. We must next assume the distance we would have the prisms from each other; let this be B b, and draw b S. This is cut by the farther edge, E F, of the first prism produced, in e; then a line joining e and D, gives f, for the corner of the second prism, from whence all the rest of it is determined in the same manner as the first: the second prism being finished, the corner g of the third prism, is determined from the section of the farthest edge of the second, with the visual b S, &c. &c. One row being finished a parallel row may be drawn in the same manner: or if a rectangle a b c d is drawn equal and even (with respect to the entering line E N) to the rectangle A B C D, then lines drawn parallel to the *Horizontal line*, from the corners of the prisms, whose representations are already found, will, by their *intersection* with the *visuals* c S, b S and d S give the corners of the prisms of the other row, as may be seen by a bare inspection of the figure.

E X A M P L E IV.

Figure 40 exhibits three rows of upright prisms, having the same dimensions as those in fig. 38: but here all the sides are oblique to the picture. H L is the *Horizontal line*,

H 4

S its

S its center, and S I its distance, all which are supposed, given or assumed. (The distance S I is laid from the Horizontal line H L *downwards* to I, in this example, for want of room; but in general it is best to use a separate paper, on which a line may be drawn to represent the *Horizontal line*, or any other *vanishing line*. The center and distance being then laid down, the *vanishing* and *dividing points* may be found in this paper agreeable to § 24, 25 and 78; and then transferred to the *vanishing line* &c. in the paper you work on.)

Also, having determined or assumed the obliquity of one of the sides of the prisms;

1st, Draw thro' the eye (I) the line I H, cutting the *Horizontal line* in H, and forming with it the angle L H I, equal to that obliquity (*Vide* § 24.)

2d, Draw I L at right angles to I H, and cutting the *Horizontal line* in L, then are the points H and L the *vanishing points* of the Horizontal boundaries of the prisms to be represented (*Vide* § 24, 25.)

3d, Let the point B be assumed for the nearest corner of the base of one of the prisms, and at this point put A B parallel to the picture and perpendicular to the *entering* or *Horizontal line*, and equal to the given height of the prisms by (prob. 2. § 68.)

4th, Through A draw a working line parallel to the *Horizontal line*, and on it lay from

from A, A C, equal to the width, and A f equal to the distance that the prisms are to be from each other, then through all these points A B C and f, let visual lines be drawn to H.

5th, On the *Horizontal line*, from H lay H D equal to H I, cutting H L in D. Also bisect the right angle L I H by the line I d.

We may now find the representation of as many prisms, or as many rows of prisms as we please, in the following manner:

By problem 4. § 78. cut off from A H, parts equal to A C, and A f, alternately; thus, a ruler laid by C and D gives the point 1 by cutting A H, through which a line drawn parallel to A f gives the point 2, by cutting f H. A ruler laid by 2 and D, gives the point 3, by cutting A H; through which a line drawn parallel to A f, gives the point 4, &c. In this manner may A H be divided *ad infinitum*, into spaces alternately equal to A C and A f. Through all these points 1, 3, 5, 7, 9, &c. let lines be drawn to L, and crossed by a line drawn from A to d, cutting the said lines drawn to L, in the points m, n, o, p, q, &c. through these points, lines drawn to H form, by their intersection with those already drawn to L, the *representations* of the tops of the prisms, and how the other parts are found may be seen by inspecting the figure.

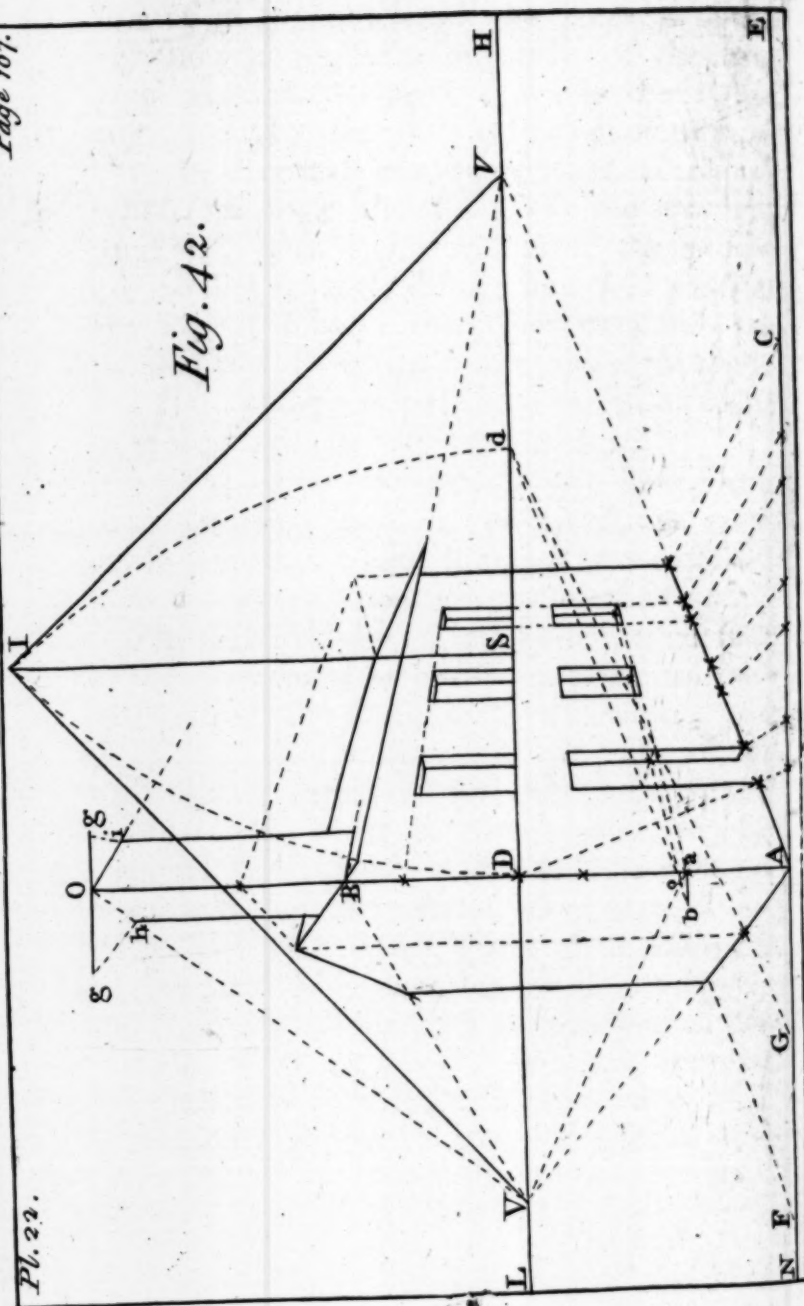
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It is needless to give a formal demonstration of all this practice: the point *d* is the vanishing point of the diagonals of the original squares, seeing *I d* bisects the right-angle *L I H*; for it is easy to demonstrate, that the diagonals of a square bisect its angles; wherefore since the sides of the original squares, which form the tops of the prisms are parallel to their *distance lines* *I H* and *I L* (by § 9) it follows that their diagonals are parallel to *I d*, which bisects the right angle *L I H*; consequently *d* is the vanishing point of those diagonals (*Vide* § 9 and 19.)

Several other methods might have been used, but the above is conceived the most exact, and at the same time not very tedious; however we will mention one method more.

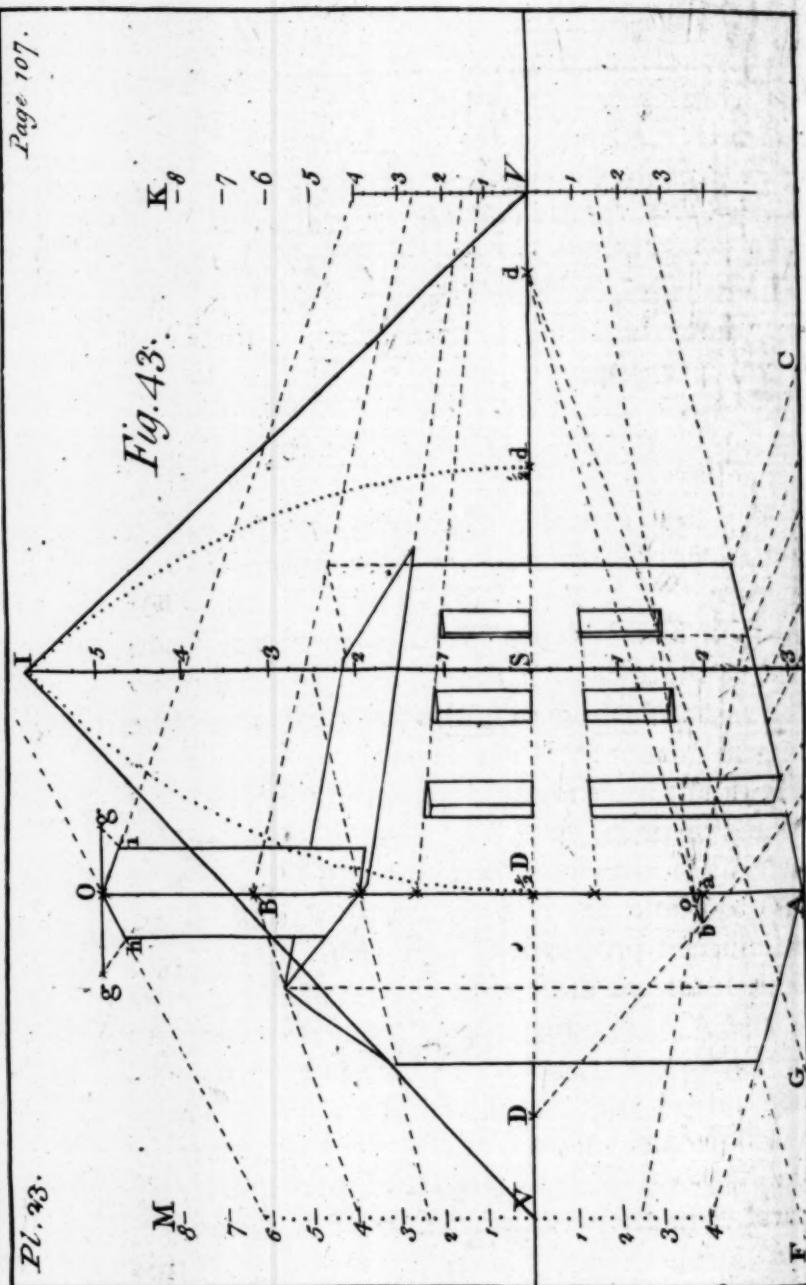
Through *A* draw an indefinite line parallel to the *Horizontal line* as before, (*Vide* fig. 40.) in which take *A C* equal to the width, and *C F* equal to the *distance* of the prisms from each other: then *C D* gives the point 1, and *F D* gives the point 3; then by means of the diagonal *A d*, finish prism II, produce its sides, and they form two sides of each of the prisms III, IV, V and VIII; then III may be compleated by means of its diagonal 3 *b*; and from hence two sides of the prisms VI and IX will be obtained. VI may next be compleated by its diagonal *A b*, and by it V may be compleated, and two sides of VII may

Fig. 42.



Pl. 23.

Fig. 43.



may be drawn, and the rest of it compleated by the diagonal of III produced. VII being compleated, the fronts of IV may be finished, and two sides of X may be drawn. X may then be finished by the diagonal of II and VI produced, and from it IX and VIII may be compleated. By this method, which is more brief than the former, we may find the *representation* of any number of prisms we please.

EXAMPLE V.

Fig. 42 contains the *representation* of a house, whose fronts are all oblique to the picture. The vanishing points *V* and *V'* of all such lines as are parallel to the horizon and to the fronts of the house are found by the same method as in the last example. The nearest corner *A B* is assumed, or it may be put in any position and of any dimensions by means of prob. 2. § 68. On this corner the heights of the door, windows, side, and roof of the house are laid according to any given or assumed proportion, and from these divisions lines are drawn to the vanishing points *V* and *V'*. Having thus found the heights of the several members, their lateral dimensions are found by prob. 4. § 78. Thus, through any point (as *A*) in the corner of the house *A B* draw an indefinite line *A E* parallel to the vanishing line *H L*; and on
this

this line set the breadths and distances of the door and windows in any proportion (assigned) to the heights before laid down, as denoted by the divisions between A and C. Next lay the distance I *V* of the vanishing point *V*, from *V* to D on the Horizontal line H L; also draw the line A *V*: then lines drawn from D to the several divisions on A C give the positions of the door, windows, &c. on the visual line A *V* as expressed by the dots; through which lines drawn parallel to A B, will, by their intersections with those before drawn to *V*, form the representation of the external parts of the left hand front of the house. Next through the point a, (being the height of the bottom of the lower windows) let a b be drawn parallel to the Horizontal line, and made of any determined length for the thickness of the wall: also through this point a, let a line be drawn to *V* and a part of it, a o, cut off equal to a b (by prob. 4. § 78) Through o draw o *V*; also through the points where a *V* cuts the farther edges of the door and windows, let lines be drawn to *V*, cutting o *V* in the points distinguished by the large dots, through which points lines drawn parallel to A B, form the representation of the inner edges of the windows, &c. which appear through the openings.

The left hand front being finished, the length of the right hand front is laid from A to F in the line A F, drawn through A, parallel

parallel to the Horizontal line $H L$, and its half to G , these are transferred to $A V$ by prob. 4. § 78. and from them the whole front is finished. Next how the roof is completed will appear from inspection. For the chimney (which I suppose to be posited in the nearest corner of the house) lay its height from B to O , thro' which $g g$ is drawn parallel to the *Horizontal line* $H L$, on this is taken $O f$ equal to the thickness of the wall, and $f g$ equal the width of the chimney: these distances are then transferred to $O h$ and $O i$ by prob. 4. § 78. How all the rest is formed will be better understood by inspecting the figure and reflecting on the theory, than by many words.

R E M A R K.

By this time, I imagine, the reader has discovered two very disagreeable circumstances in the practice of this art; the first is, that if he uses a short distance, his representations appear deformed and unnatural when seen at that distance which people will naturally view them from (see page 83.) as may be seen in the examples we have hitherto described, which the smallness of the plates rendered unavoidable. The second difficulty is, that if a long distance is used as recommended in page 84 and 85, the *vanishing points* of such lines as are nearly parallel to the picture

ture, and the *dividing centers* of those which are nearly perpendicular to it, fall so very remote from its center, that they frequently exceed the bounds of the paper we draw on, and require a long ruler to join points that are so far apart; the using of which is troublesome and awkward. These inconveniences we hope entirely to remove by the three succeeding problems.

P R O B L E M V.

LXXX. At any point A (fig. 37.) in a given *visual line* A V tending to any *vanishing point* within the picture, to cut off a part A c that shall represent an *original line* of a given length, when there is not room to lay down the whole *distance* V L of the said visual A V.

1st, By problem 1. § 65. page 48. at the given point A put A F parallel to the *vanishing line*, and equal in representation to the given length.

2d, Instead of laying the whole distance from V to L, as taught in prob. 4. page 95. § 78. lay one half, one third, one fourth, or any part of it from V to D, (in the figure V D is one third of V L.)

3d, Take A f equal to the same part of A F which V D is of V L (in this case A f must be one third of A F) and draw f D, which cuts the given visual A V in (c) the same

same point in which it is cut by F L, which is drawn according to the method before taught in prob. 4. § 78.

The truth of which is evident from E 6. 10. method 4, and § 77.

EXAMPLE VI.

In the third example I was obliged to make use of a very short distance, which made the prisms afford an *unnatural appearance*: wherefore in fig. 39. I have given another *representation* of the same prisms. I lay the distance S D on the *horizontal line*, the same as before; but I suppose S D to be only one fourth of the true distance: then on B C, I take B a, equal to one fourth of the width of the prisms; and B b, equal one fourth of their distance from each other, and draw the visuals a S and b S; then prob. 5. and a perusal of the figure, will sufficiently explain the rest of the operation.

PROBLEM VI.

LXXXI. Any vanishing line S V (*Vide* fig. 44.) its center S, and any part S i, as one half, one third, one fourth, &c. of its distance being given, through a given point A, to draw the *visual* A B, of an *original line*, making a given angle with the *entering line* E N; and to cut off a part (A B) from the
given

given point A, equal in *representation* to any assigned length.

S O L U T I O N.

In fig. 44. we suppose S i to be given equal to one third of the *true distance* of the eye.

1st, Through i draw i v, making the angle i v S equal to the angle, which the *original line*, you want to represent, makes with the *entering line*.

2d, Through the given point A, draw the *occult lines* A v and A S.

3d, Through any point r, taken at pleasure in A S, draw r t, parallel to the *entering line* E N, and cutting A v in f.

4th, On this line (r t) repeat the distance r f three times from r to t (because S i is one third of the true distance,) then a line joining A and t will be the visual required, which if produced tends to V its true vanishing point, which may be inaccessible, or exceed the bounds of the paper.

To perform the latter part of this problem,

1st, Assume or take i m equal to one third of the length you want A B to represent (because, as before, S i is one third of the true distance.)

2d, Thro' m draw m n parallel to the *vanishing line* S V, and cutting i S in n.

3d, From S lay i S (the given one third of the distance) to D and from A lay A o parallel

parallel to the vanishing line, and equal to $i n$.

4th, Join $o D$ cutting $A S$ in b ; then $b B$ drawn parallel to the *vanishing line*, $V S D$, cuts $A B$ in B . and $A B$ is the representation of an original line parallel to $i m$, (*Vide* § 32) and of 3 times its length. Q. E. I.

DEMONSTRATION.

It is plain from the similar triangles $I S V$ and $i S v$ fig. 44. that because I (the true distance of the given *vanishing line*) is 3 times as far from S , as i , is; V (the true *vanishing point* of a line inclined to the *entering line* in the angle $I V S$ or its equal $i v S$) is 3 times as far from S as v is. (E 6. 4.) Now since $r t$ is parallel to the *vanishing line* (*by construction*) and equal to 3 times $r f$, therefore if lines drawn from A thro' r and f tend to the points S and v , (as they do by construction) then $t A$ must, if produced tend to a point in the *vanishing line* $V S$, three times as far from S as v is, (by E 6. 4.) or in other words, it must tend to the point V , as was intended.

Again; it appears from this construction, that $A B$ is the *representation* of an *original line* parallel to $I V$ (*Vide* § 32) and consequently to $i v$: also $A b$ is the *representation* of an *original line* parallel to $S i$: and $B b$ is the *representation* of an *original line* parallel to $v S$ (*Vide* § 44) or $m n$: wherefore $A B b$ is the

I

repre-

representation of an *original triangle* similar to imn , (by E 6. 4.) But Ab is the *representation* of a line equal to AO or ni , at the distance SD or Si , (by § 76. page 93.) consequently it is the *representation* of a line equal to 3 times ni , at the *true distance* of the eye IS , (by § 80.) whence AB is the *representation* of a line equal to 3 times im , at the *true distance* IS . Q. E. I.

This problem is of great service, when we want to find the *representation* of a single line, which tends to a remote *vanishing point*, and which, as it sometimes happens, will facilitate our finding the *representation* of several other lines. But if we want to find the *representations* of many lines which tend to some one *vanishing point* lying beyond the bounds of our paper, the following method will be productive of much more dispatch.

Another Solution of the last Problem.

LXXXII. 1st. Having laid down iS equal to one third, one fourth, &c. of the true distance IS , (fig. 45.) of the *vanishing line* SV , draw, as before, iv , making with the *vanishing line* the angle ivS , equal to that which the *original line* to be represented makes with the *entering line*.

2^d. Produce iS both ways, and through v draw a line parallel to it.

3^d.

3d. From S take any small distance in your compasses, and repeat it each way from S; and let these divisions be numbered as the figure shews.

4th. Take in your compasses as many of these divisions as is one less than the number of times that S i is contained in the *true distance* S I. Thus in the present case S I being 3 times S i, I take 2 divisions from the line S i, and lay them on its parallel drawn through V, as often as I please.

Lastly, divide each of the divisions laid from V into so many parts as expresses how often S i is contained in S I, (in the present case 3) and let these be numbered as the figure shews. Now if we would draw thro' any point (A) in the picture, a line tending to the proper *vanishing point* V, we need only lay a ruler by the proposed point A, and any two correspondent divisions on the divided lines; because any line passing through any two correspondent divisions tends to V, the *true vanishing point*.

DEMONSTRATION.

Through I (fig. 45) the *true distance* or *place of the eye*, draw IV parallel to i v; then is V the *true vanishing point* of the original line to be represented (by § 32) and it is plain (from E 6. 4.) that if S I is divided into equal parts, lines drawn to V from those

I 2
divisions,

divisions, will divide $v F$ into the same number of equal parts; and that these parts will be to each other, as $V S$ to $\bar{V} v$ or as $S I$ to $I i$, or as the number which shews how often $S i$ is contained in $S I$, to that number less one. In the scheme before us we have $S I : I i :: 3 : 2$, wherefore a division on $S I$ ought to be to a division on $v F$ as 3 to 2. Consequently 2 divisions on $I S$ is equal to 3 on $v F$, and this being made so in the construction, it follows that lines drawn through any correspondent divisions on the lines $S I$ and $v F$ will meet in V . Q. E. D.

LXXXIII. *We come now to the second part of the Problem; wherein we shall shew how the center S , (Vide fig. 45.) of a vanishing line may be used instead of the dividing center (Vide § 47) of any visual line, which tends to an inaccessible vanishing point.*

Let $A V$ (fig. 45) be the given *visual*, and from A , it is required to cut off a part ($A B$), which shall represent a line of a given magnitude.

1st. From v (found as before-mentioned) lay $v d$ on the *vanishing line* equal to $v i$.

2d. Through d and S draw any two lines $d C$ and $S c$ parallel to each other; make $d C$ equal to the length of the line required to be represented: and lay a ruler by v and C cutting $S c$ in c . Lay this line $S c$ on the
entering

entering line (E N,) and by prob. 1. (page 48) at A put a line A e equal and parallel to it.

Lastly, A ruler by e and S cuts A B in B, then is A B the line sought.

DEMONSTRATION.

The reason of this operation will be plain, if we reflect on what was said in § 77. page 95. where it appears that we may use any part as S V of the *true distance* (V D or V I) of any visual A V, if we shorten the true length of the line we want to represent in the same proportion, that is, if instead of using the true distance V D, we would only use a part of it as V S, we must make A e (by prob. 1.) to represent a line S c parallel to the picture, which bears the same proportion to d C, the true length we want to represent, as S V the distance we use, is to V D or V I, the *true distance*; that is, we must make $V D : V S :: d C : S c$; let us now examine our figure, and we shall find this is actually performed.

Draw I D and i d, then in the similar triangles i S d and I S D we have $S I : S i :: S D : S d$. Also in the similar triangles V I S and v i S we have $S I : S i :: S V : S v$. wherefore (by E 5. 11) $S D : S d :: S V : S v$. consequently (by E 5. 18.) $S D + S V (= V D) : V S :: S d + S v (= v d) : v S$. but by construction $v d : v S :: d C : S c$. consequently $V D : V S :: d C : S c$.

I 3

from

from whence it appears that, by making A e equal in representation to S c, and joining S e, it cuts the visual A B, in the same point B, where it would have been cut if A e had been made equal in *representation* to d C, and a line drawn joining e and D. (*Vide* § 77. page 95.)

EXAMPLE VII.

The prisms represented in figure 40 appear distorted, when viewed at a *natural distance*, (*Vide* page 84 & 85) because the distance I S, which was used to find their representation is too short. We have therefore in fig. 41 represented the same prisms as they would appear if the eye is removed six times as far from the picture, as it was supposed in fig. 40. To effect this, and at the same time not exceed the limits of the plate,

1st. Draw H L for the horizontal line, assume S for its center, and draw the perpendicular S I, making S I equal to S I in fig. 40; which will now be only one sixth of the *true distance of the eye*. Also draw the distance lines I L and I H, the same as in fig. 40.

2d. Take any small opening in the compasses, and repeat it on the line S I from S, as often as shall be thought needful, and number them 1, 2, 3, &c. (*Vide* § 82. page 115.)

3d.

3d. Through the vanishing points H and L, draw lines parallel to I S, on each of which lay five parts of I S, from H to 6, and from L to 6, and divide these spaces into 6 equal parts; and let them be numbered and continued as far as necessary. (*Vide* § 82. page 115.)

N. B. *The spaces thus numbered on the lines S I, H E and L N may be subdivided into four parts each, exact enough, by the eye; as is shewn by the small dots.*

4th. Draw I b bisecting the angle L I H, and make S D equal to six times S b, (*for as b is the vanishing point of the diagonals of the squares, when the distance is S I, it is plain, the true distance being six times S I, that the true diagonal will fall six times as far from S, as the point b does.*)

We now proceed in all respects as we did in fig. 40, taking care that all lines, which tend to H in fig. 40, pass through corresponding divisions on the lines S I and H E in figure 41. and that all vanishing lines, which, in fig. 40 tend to L, in fig. 41 must pass through corresponding divisions in the lines S I and L N.

To find the points 1 and 3, (fig. 41) we have recourse to the directions given in the last problem (§ 83. page 116.) Because the true distance of all visual lines tending to H, would fall 6 times as far from S as d does

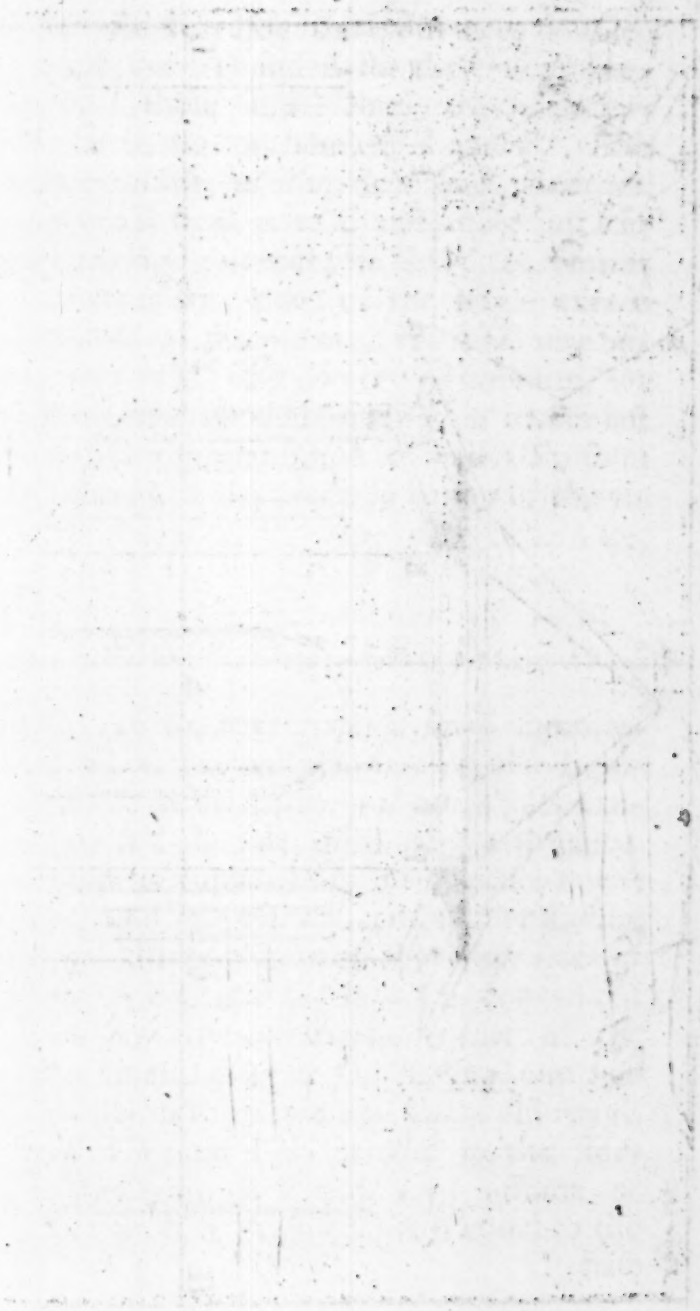
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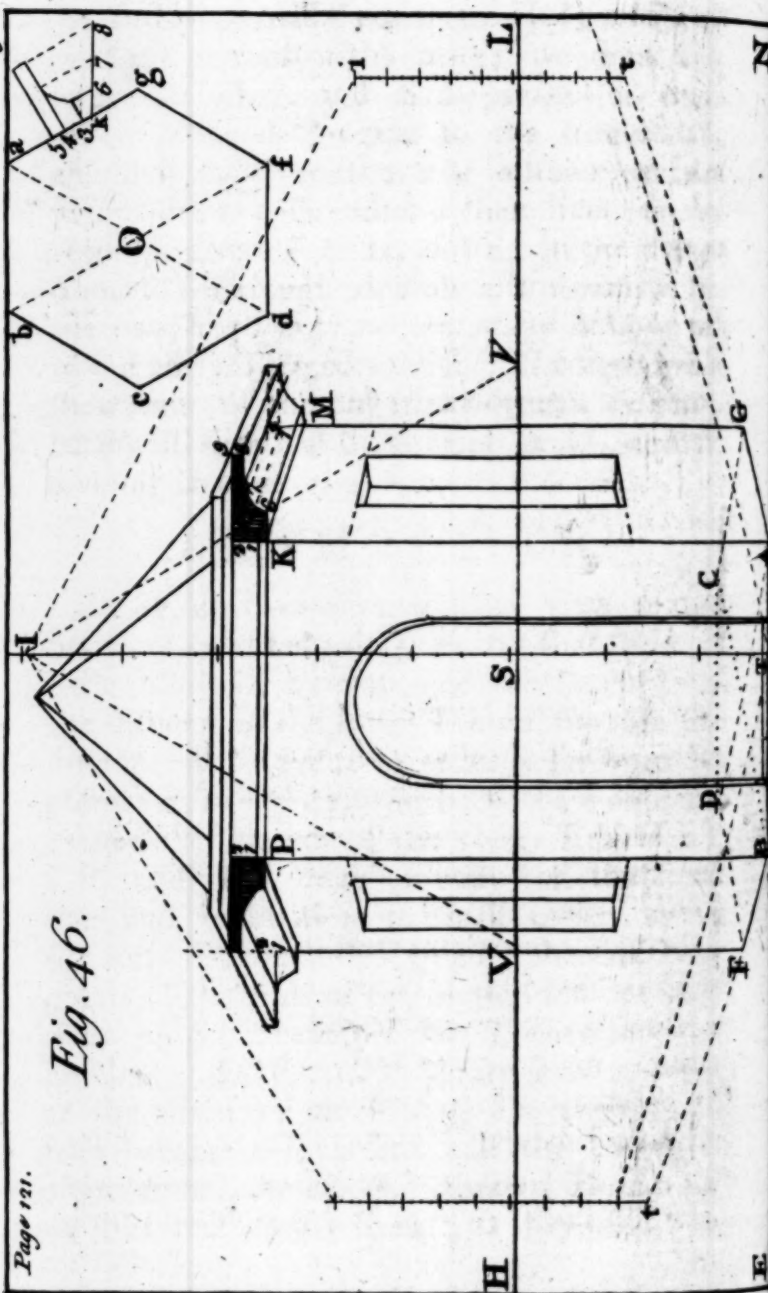
(H d

(H d being made equal to H I) and exceed the bounds of the plate; we draw any two lines d F and S f parallel to each other, make d C equal to the true width, and d F equal to the true distance of the prisms from each other; then lines drawn through C and F to H, cut S f in the points c and f. Through A draw a line parallel to the *vanishing line* H L: on it lay A C equal to S c and C f equal to S f. The figure will shew how all the rest is performed, remembering all diagonal lines tend to D, and all dividing lines to S.

E X A M P L E VII.

In fig. 43, we have another view of the house represented in fig. 42, as that seems a little distorted, by reason of the shortness of the distance of the eye. I here suppose the distance of the eye to be twice as far from the picture as in fig. 42, and because I would not exceed the bounds of the plate, I draw S I I V and I V here exactly of the same size and inclination to each other as in fig. 42. But because V and V, the vanishing points of the sides of the house, are but half their proper distance from S, therefore the divisions on V K and V M are half as large as the divisions on I S (§ 82. page 115.) Also because S I, is but half the length it ought to be, the *dividing centers* $\frac{1}{2}$ D and $\frac{1}{2}$ d are but half so far from S as they ought to be,





be, for which reason these distances doubled give the points D and d for the *true dividing centers* of those visual lines whose correspondents in fig. 42 tend to V and *V*. Also the true widths of the door and windows, &c. are laid from A to C and to F, just the same as in fig. 42 : because altho' the vanishing points of the sides of the house exceed the bounds of the picture, yet their true *dividing centers* D and d do not : if they had, we must have used the same method as in the last example, by shortening A C and A F, so as to have made S the dividing center of the visuals A *V* and A V. (See page 116. § 83, &c.)

EXAMPLE IX.

In fig. 46 we have given a small summer-house, whose ground plan is a regular hexagon. H L is the *horizontal line*, S its center, and S I is one third of its distance. One front is supposed to be parallel to the picture ; and in order to facilitate the finding of the vanishing points of the other sides of the base, A B C D F G, it will be convenient to draw in any indifferent corner of the picture a small hexagon O, having one side a b parallel to the *horizontal line*. Through I, draw I V and I *V* parallel to the sides of the hexagon, b c and a g, assume or make (by prob. 1. page 48.) A B equal to the
base

base of the front side. Produce IS to T in the entering line EN , or beyond it if you have room: then (*by prob. 6. page 112*) through the points I and T , draw lines Ii , Tt tending to a point three times as remote from S as V or V' is. Also draw ti and ti , (through the points i and i taken at pleasure in the line Ii) parallel to IT and divide these three lines into the same number of equal parts more or less, as you would be more or less exact: then (*by page 115.*) any line drawn through corresponding divisions of these divided lines will represent lines parallel to the horizon, and to the sides of the house. A perusal of the figure will shew how all the rest is performed. But the cornish may demand a little more explanation. On either side, as a g , of the small hexagon O , draw the heights and projections of the several parts of the cornish proposed to be represented in their proper dimensions, and under it draw a line parallel to the horisoptal line. Also through the projections of this cornish 3, 4 and 5, draw lines parallel to a g , cutting the last drawn line in the points 6, 7 and 8.

2d. On the corner of the house AK , lay the heights of the cornish; but instead of the true projections, take them from the line $k8$, and by this means we get the figure $K268$, which is the representation of the end of the cornish of the side $AGKM$, cut off

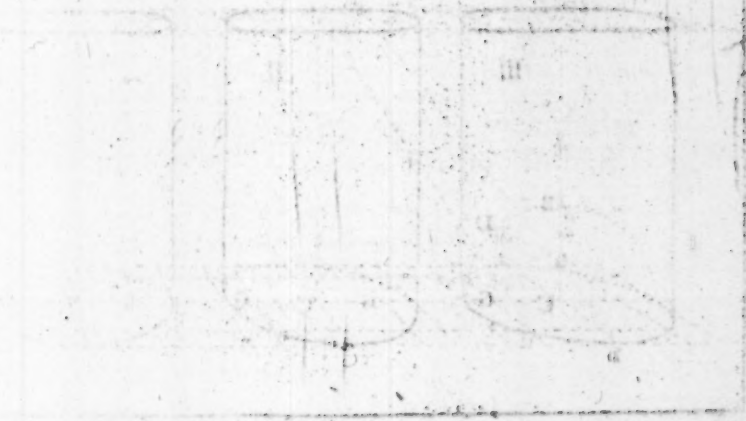


Fig. 47.

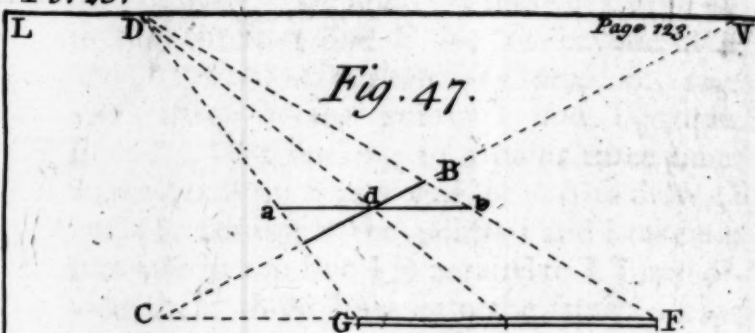
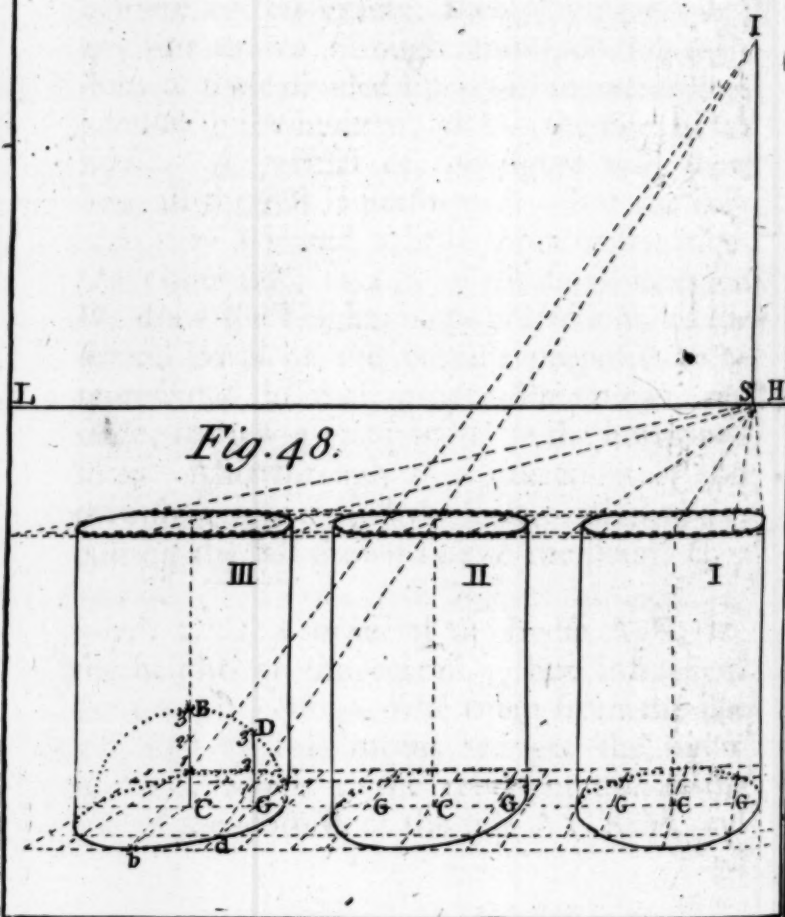
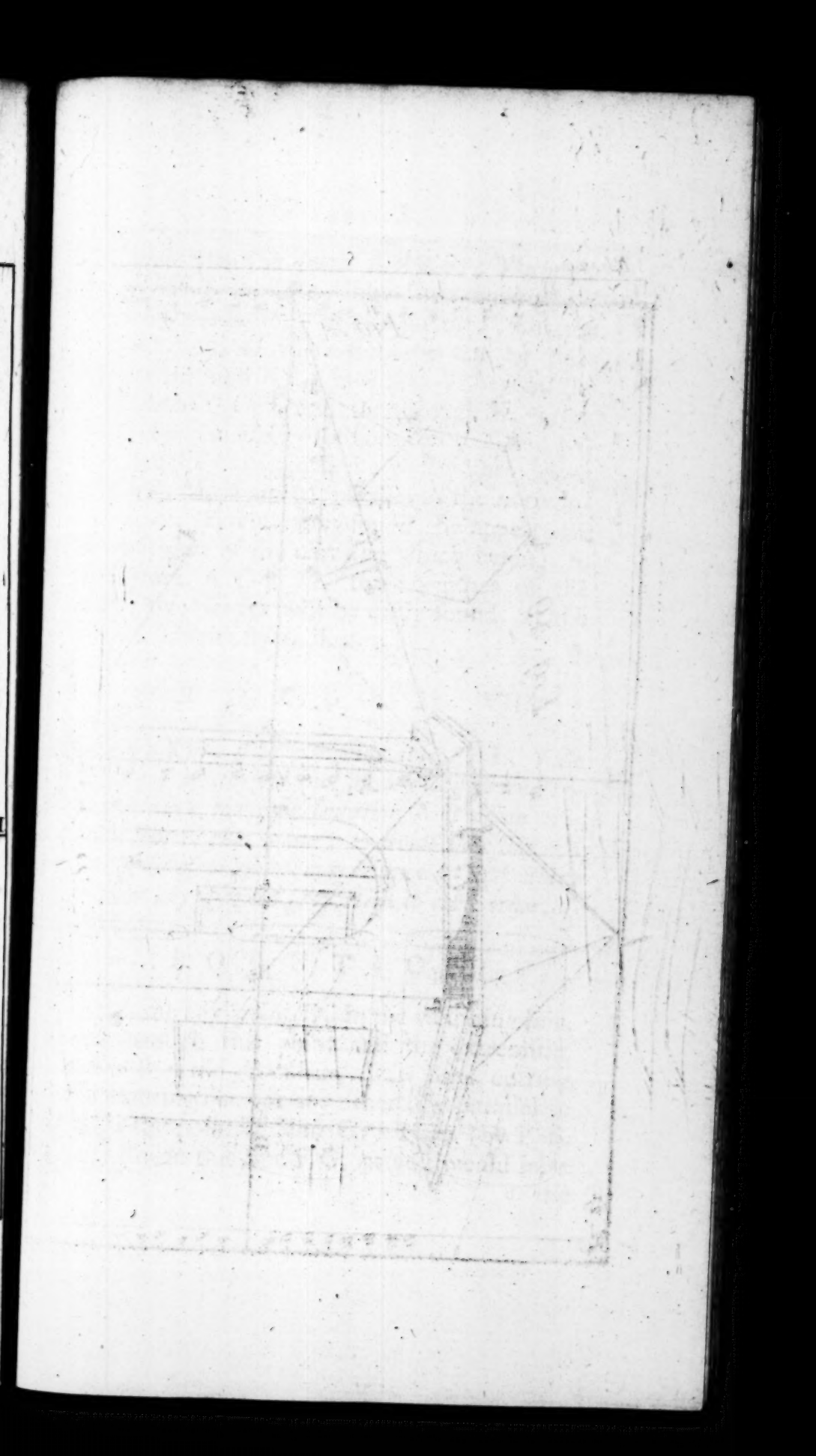
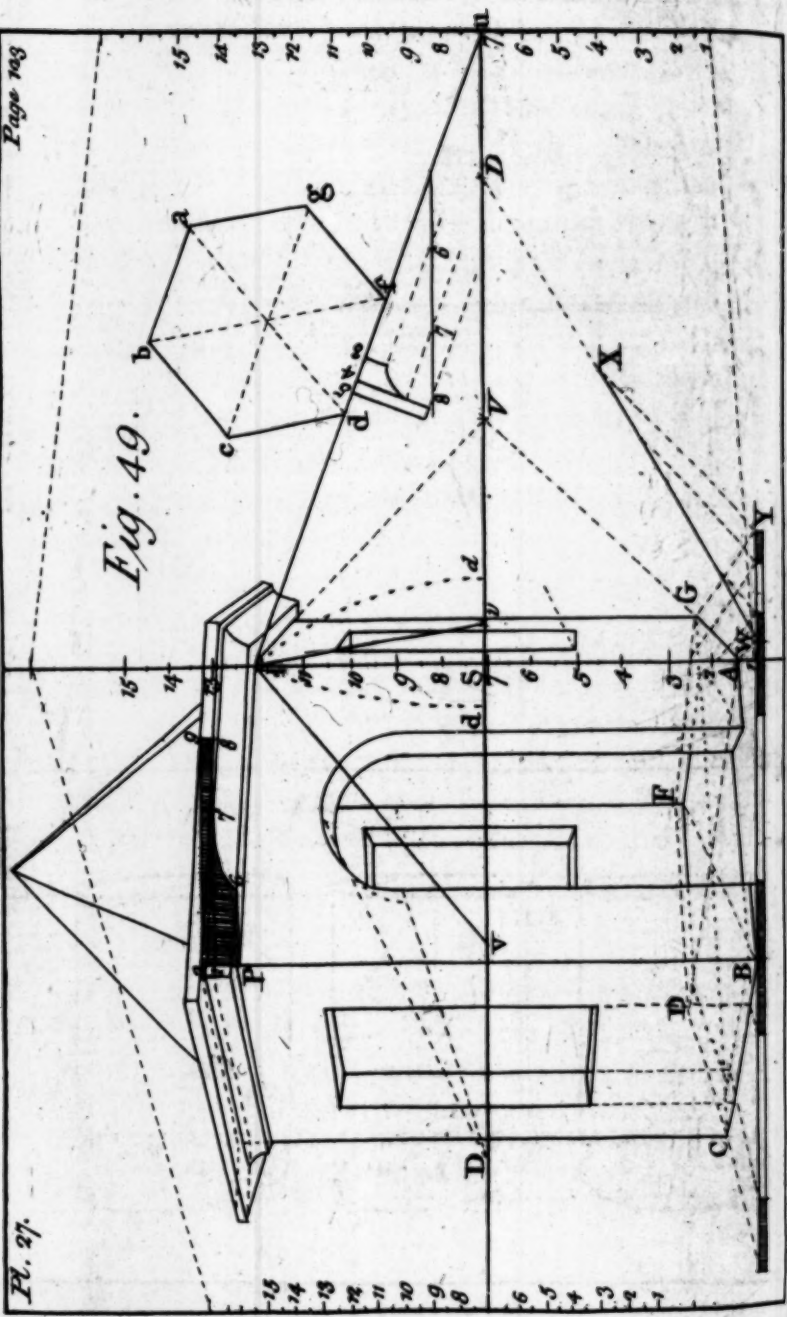


Fig. 48.







off even with the front A B K P. Through the points 6, 7, 8, 9, draw lines tending to V beyond the bounds of the picture. Let the nearest ends of these lines be cut by lines drawn through K, 1 and 2 to V beyond the bounds of the picture, (or through K, 1, 2, and corresponding divisions on I T and t i) and let the farther ends be cut by lines drawn through M, 1 and 2, parallel to the horizontal line. Having thus found the appearance of that part of the cornish, which belongs to the front A G K M, the cornishes of the other two fronts will be easily found, as the figure sufficiently indicates.

P R O B L E M. VII.

LXXXIV. *The vanishing line (V L, Vide fig. 47.) of any original plane being given, together with the representation A B of an original line in that plane; to divide that line into parts, whose originals may be equal to each other, or bear any assigned proportion to each other.*

S O L U T I O N.

Assume any point D, in the vanishing line, and through this point and the extremities A and B of the given line, draw lines cutting the entering line (or any other line parallel to it) in the points F and G. Then (by E 6. 10.) divide this line F G, as you would have
the

the given line divided, and through each division draw a line to D; then will these lines divide A B, so that its parts will be similar in *representation* to the parts of F G.

DEMONSTRATION.

Produce A B, till it cuts F G in C, then is B C F the *representation* of a triangle, whereof the sides represented by B C and F C are cut by lines parallel to the base, represented by B F, for the originals of the lines that are drawn to D are parallel to each other (by § 17.) therefore the original of A B is divided in a manner similar to F G. (by E 6. 10.) Q. E. D.

R E M A R K.

This problem is often of use, when we have found the representation of a line, and we want to divide it in any proportion, but cannot have recourse to its dividing center, as we shall have occasion to shew in the next example.

E X A M P L E X.

In fig. 49, we have the same summer-house as in the last example, but now that side, which before was parallel, is made a little oblique to the picture. I S is one sixth part of the true distance of the horizontal line. The small hexagon is drawn as in the last

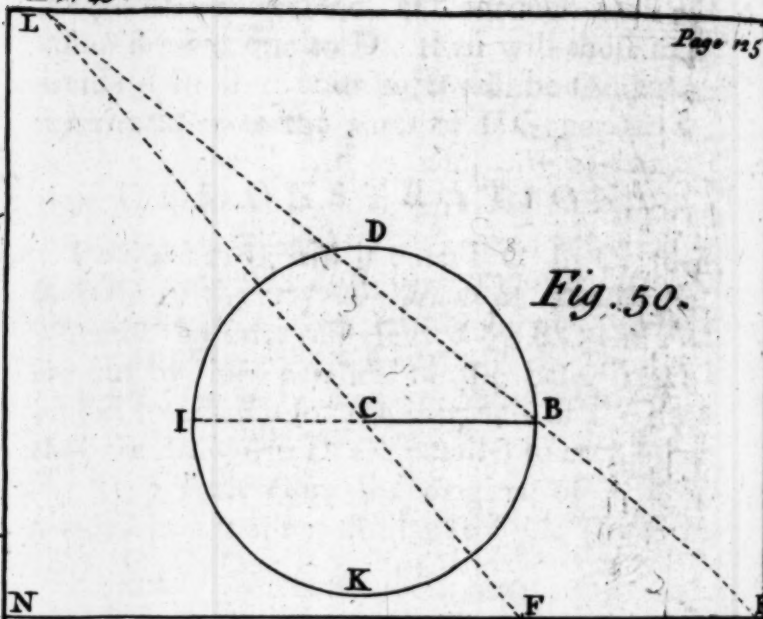


Fig. 50.

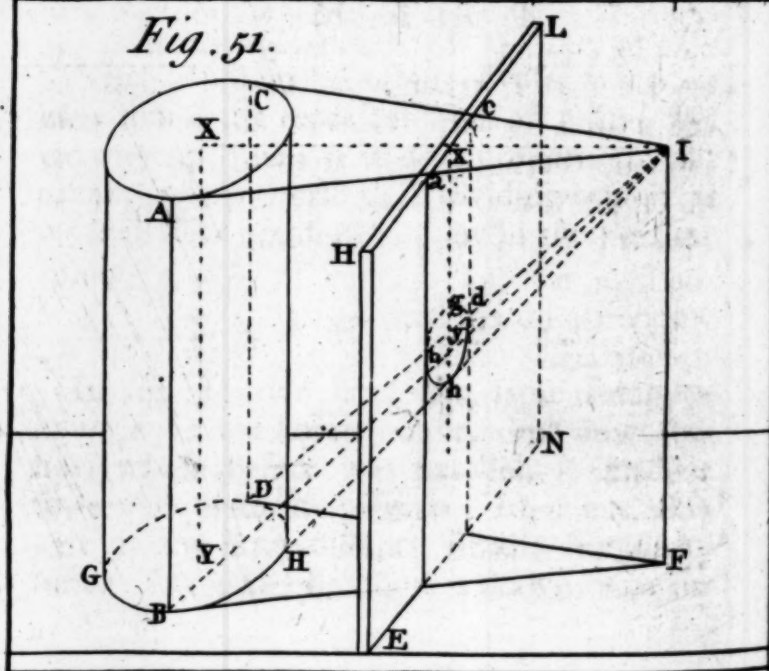


Fig. 51.

last example, Having now the sides $a b$ and $d f$ inclined so much to the horizontal line, as we would have the side $A B$ inclined to the entering line, lines are drawn through I parallel to the sides of the hexagon: these give the points v , v and u , (each of which is only one sixth as far from S as it ought to be) $S v$ repeated 6 times gives V for the vanishing point of the side $A G$, and of all lines, whose originals are parallel to it. $S d$ repeated 6 times from S , gives the point D for the *distance* or dividing center of $A B$ and its parallels, and $S d$ repeated 6 times gives D for the distance of $B C$ and its parallels; but as the vanishing points of these lines far exceed the bounds of our plate, we have recourse to prob. 6. page 114 and 115. B may now be assumed for the nearest corner of the base, and $B P$ taken equal to the height of the nearest corner. How all the rest is performed, the figure will shew. But it may be proper to say, that the representation of the base $A B C D F G$ is first found by drawing the sides to tend to their vanishing points, and cutting them by the diagonals which tend to the same vanishing points, because the diagonals of a hexagon are parallel to its sides.

The situation of the door and window on the sides $A B$ and $B C$ are found by their proper distance points or dividing centers D and D . But we must proceed in another manner to divide $A G$, because its true dividing

viding center falls beyond the bounds of the picture, for which reason we make use of problem 7. page 123. *viz.* from any point as D in the horizontal line H L, we draw lines through A and G cutting the scale line in W and Y. Through W we draw any line W X at pleasure, and make it equal to the true length of the side A G, and on this line the breadth and situation of the window is laid. Lay a ruler by the points X and Y, and draw lines through the divisions on W X parallel to X Y, and cutting W Y similar to it, (*Vide* E 6. 10.) then a ruler laid by D and the several divisions on W Y gives the situation and width of the window on A G, from whence its whole representation is found. The cornish is found in the same manner as in fig. 44. As the door in this and the last example was arched, the reader may be supposed ignorant of the manner by which its *representation* is found, till he is master of the two next problems.

P R O B L E M VIII.

LXXXV. *The entering and vanishing line, E N and H L (fig. 50.) of any original plane being given. At any point C, in that plane, as a center, to draw the representation of a circle situated in a plane parallel to the picture, and whose radius is given; as E F.*

By

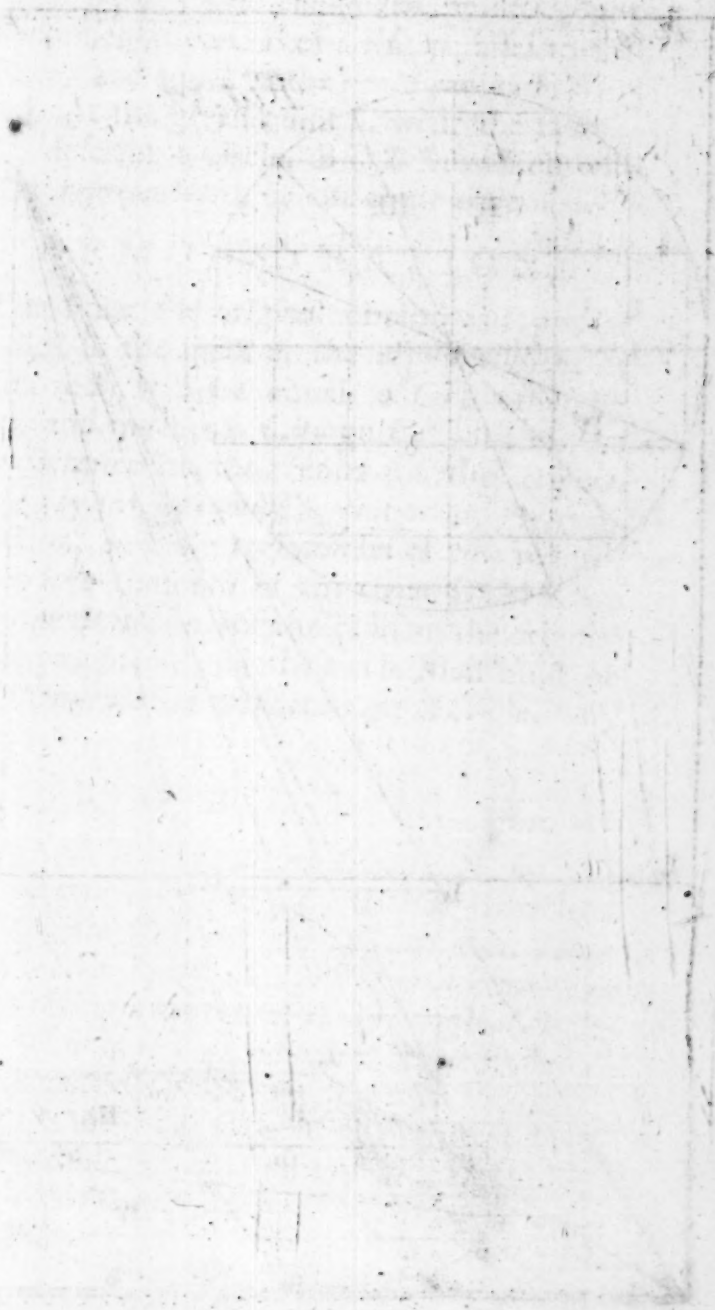
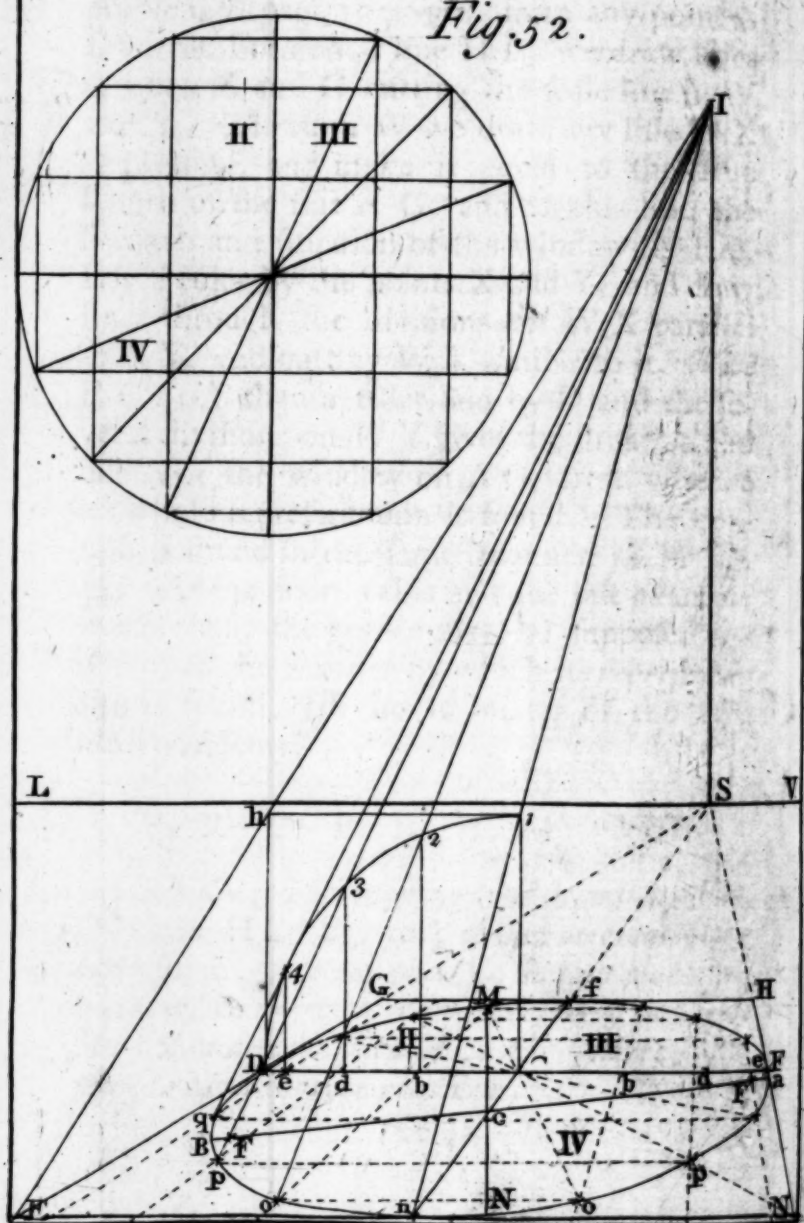


Fig. 52.



By § 65 or 68 at the given point C, put the representation C B of a line parallel to the picture, and equal to the given radius E F.

2d. At the given point C with the radius, B C, describe a circle, B D I K, which will be the *representation* of the circle proposed.

DEMONSTRATION.

For since the original circle was supposed parallel to the picture, the *representations* of all its *radii* will be equal to each other by § 79, and must pass through C, which is the *representation* of the center of the *original* circle (*by the supposition*); but C B, by construction, is the *representation* of one radius; wherefore the *radii* of the circle B D I K are the *representations* of the *radii* of the original circle proposed; and the circle itself must be the *representation* of that original circle. Q. E. D.

P R O B L E M IX.

LXXXVI. *The entering Line (E N, fig. 52) and the vanishing line V L, with its center (S) and distance (S I) of any original plane being given; at any point (C) given as a center, to draw the representation of any original circle, (situated in that original plane) whose radius is of a known magnitude.*

There are many ways to solve this problem but the most general and expeditious I apprehend to be the following.

1st.

1st. At the point C (*by problem 1.*) put CD parallel to the entering line and equal to the radius of the circle proposed, also draw C I perpendicular to C D; and describe the quadrant C I D.

2d. Through any points b, d, e, &c. taken at pleasure in the radius CD, draw lines parallel to C I, cutting the periphery of the quadrant in the points, 2, 3, 4, &c. also through the same points draw lines C n, b o, d p, e q, &c. to the center S of the vanishing line; then a ruler laid by the place of the eye (I,) and the points 1, 2, 3, 4, &c. gives the points n, o, p, q, &c. through which points, draw a curve, which will be the representation of one quadrant of the circle proposed, from whence the other 3 may be easily determined thus. Produce C D to F, making C F equal to C D and lay the same divisions between C and F as were before laid between C and D. Through these divisions draw the lines b S, d S, e S and F S, and let these be crossed by lines drawn through n, o, p, q, parallel to the entering line, which will give the points through which the curve of the quadrant marked IV must pass. Lay a ruler by the points n, o, p, q in the quadrant first found, and the center C, and its intersection with the lines b S, d S, &c. gives the points to form the curve of quadrant II. from which the quadrant marked III is determined as the figure shews by inspection.

By

By the last problem it appears, that $C I D$ is the representation of a quadrant parallel to the picture and having the same center and radius with the circle to be represented, and $C I$ is the representation of a radius and $b 2$ $d 3$ $e 4$ &c. of chords perpendicular to the radius $C D$. But $C n$ is also the representation of a radius, and $b o$, $d p$, $e q$, &c. of chords perpendicular to the same radius $C D$, and cutting it the same points, wherefore $C n$ should be represented equal to $C I$, $b o$ equal to $b 2$, $d p$ equal to $d 3$, &c. but it is evident from prob. 4, Cor. 3. § 79. that $C n$ is equal in representation to $C I$; $b o$ to $b 2$; $d p$ to $d 3$ and $D E$ to $D h$, &c. wherefore $D q p o n C$ is the perspective representation of a quadrant equal to that represented by $C I D$. A little reflection on the nature of lines drawn in a circle parallel and perpendicular to each other, will fully shew the reason of the method laid down for finding the other three quadrants; for which reason a circle is drawn in the upper part of the plate, and crossed with the lines made use of to determine its representation.

In this example 16 points are found to determine the path of the *ellipsis* which represents the proposed circle, which is amply sufficient for most purposes; but if greater exactness should be thought necessary, another line drawn in the quadrant $C I D$ would have given 20 points. For every line

so drawn gives 4 points: but in most cases a much fewer number of points will be found sufficient to determine the shape of the *ellipses*. And if it is to represent the base of a small column the ellipse may be determined by the circumscribing square E N G H alone.

EXAMPLE XI.

Fig. 48 is the *representation* of three cylinders; the centers C, C, C of whose bases are supposed to be situated in a line parallel to the picture, and equally distant from each other. H L is the horizontal line, S its center, and S I is made equal to one third of its distance, because we have not room to lay down the whole; for which reason the radius B C, and the chord D G (which as the radius is small, are sufficient to determine the representation of the circle) are divided into three equal parts, and lines drawn through I I and I I give the points b and d; (*Vide* § 80. page 110.) The remainder of the *representation* of the base is found as the last problem directs. The *representation* of the base of that cylinder (marked III) which is most remote from the center S of the *horizontal line* being found, the other bases are easily determined from it, as the figura sufficiently shews. Through the centers C, C and C, lines drawn perpendicular to the horizontal line, representing the

axis

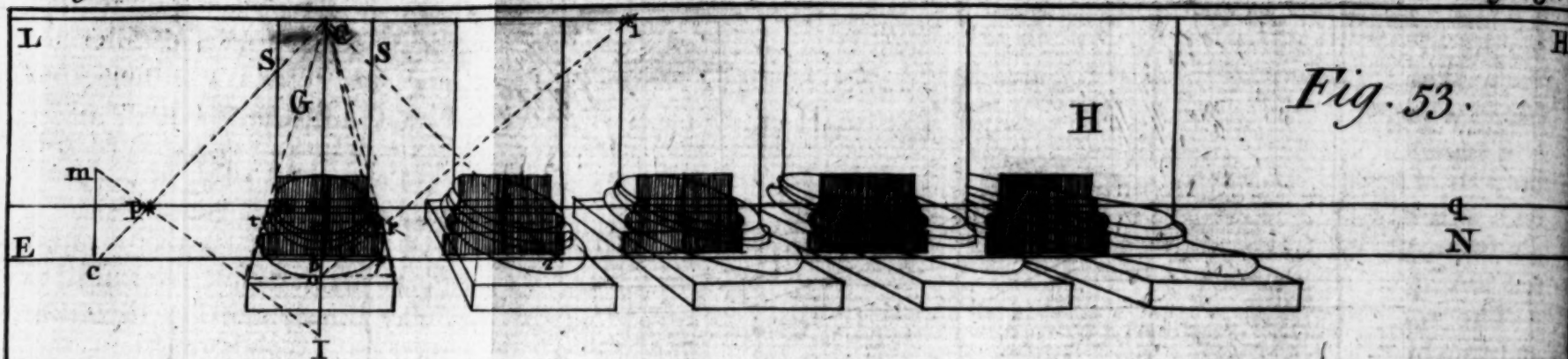


Fig. 53.

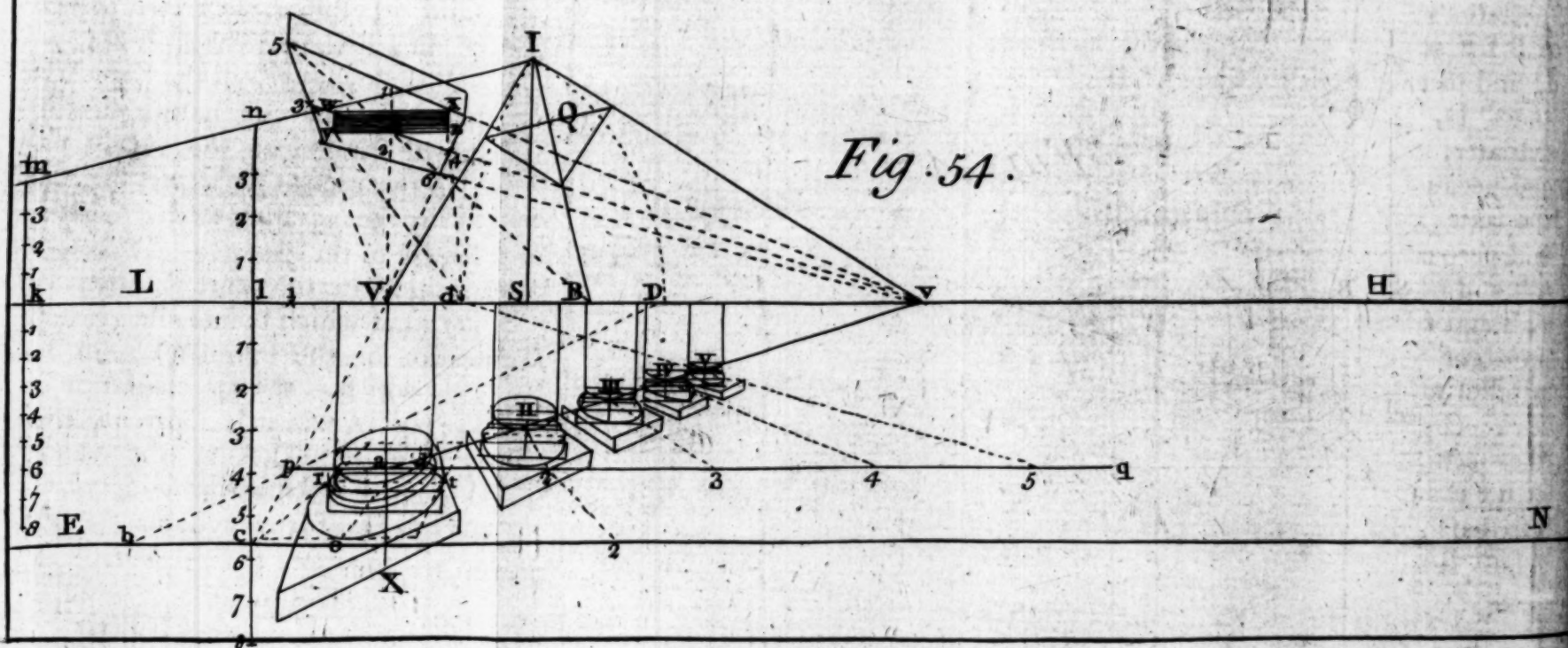


Fig. 54.

axis of the cylinders, which (by prob. 2. § 68) are made equal in representation to the height of the cylinders. Also parallel to these axes, lines are drawn touching the *ellipses* which represent the bases; these lines will represent the extremities of the apparent breadth of the cylinders. Lastly, the top of each axis being considered as a center, circles are formed (by the last prob.) whose diameters are equal to the diameters of the bases.

The reason why lines parallel to the axes, and touching the *ellipses* which represent the base, are the extremities of the apparent breadth of the cylinders, will soon appear if (fig. 50.) we suppose I, to be the eye of a spectator viewing a cylinder, and two planes A B I F and C D I F to pass through the eye I, and to touch the cylinder, in the lines A B and C D, these lines, by the nature of the cylinder, must be parallel to the axis X Y; and because, in the present case, we suppose the axis X Y to be parallel to the picture, and perpendicular to the ground. Therefore a b and c d, which are the representations of A B and C D, (and which bound the greatest apparent breadth of the cylinder) must be parallel to x y, which is the representation of X Y. (by § 53.) Again it is apparent, that the sections B F and D F of the planes A B I F and C D I F with the ground, touch the base B D G H of the cylinder in the points B and D, therefore the *perspective re-*

K 2

presentation

presentation of these lines BF and DF must touch the *representation* $b d h g$ of the base: but the *representation* of BF must lie in the line $a b$ produced; for AB and BF being situated in the same *visual plane* ($ABIF$) passing through the eye: their *representation* must lie in the line $a b$, which is the section of that plane with the picture, (by § 16.) wherefore $a b$, the *representation* of the *apparent extremity* of the cylinder is parallel to xy , and touches the ellipsis $b d g h$, which is the *representation* of the base $BDGH$ of the *original cylinder*. The same may be demonstrated of cd , which is the *representation* of the other *apparent extremity* of the cylinder.

EXAMPLE XII.

In figure 53, there is represented a row of equidistant columns situated in a line parallel to the picture; the plan of which, together with the positions of the eye I , and the picture PP , are laid down in fig. 32.

Draw HL (fig. 53) at pleasure for the *Horizontal line*, assume the point C for its center, and draw the perpendicular IC equal to IC in fig. 32; then is the point I (fig. 53) the place of the eye.

By fig. 32, it appears that the center (m) of the nearest column (G) is situated in the line IC drawn through the eye perpendicular to the picture PP ; wherefore the line IC (fig.

fig. 53,) is the representation of the axis of the first column.

At such a distance below the horizontal line as I would have the tops of the bases of the columns below the eye, I draw the line EN (fig. 53,) which is the entering line of a plane passing through the tops of the bases parallel to the ground. At any point as e in this line (EN) I draw em perpendicular to it and equal to Cm in fig. 32; then I draw eC , and mI cutting eC in p . (Vide § 79. Cor. 3. Prob. 4.) Then the line pq drawn through p parallel to the horizontal line, will represent the original line which passes thro' the centers of the tops of the bases of the columns: but pq cuts IC in a wherefore a is the representation of the center of the top of the base of the first column.

From the point (b) where the entering line EN cuts the perpendicular IC , I lay $b1$ = the true radius of the bottom of the columns, and $b2$ = the distance which the centers of the columns are from each other: then drawing lines from the points 1 and 2 to C , they cut pq in the points d and 2 .

a is the representation of the lower radius of the first column; (Vide prob. 1. § 66.) proportionable to which, I draw the several members of the base, according to the measures assigned by the architects*, on each side the

K 3

* In these examples I use the Tuscan order, because its members are more simple and large than any other.

axis I C. And from the center C, I lay on the horizontal line the distance $Ci = CI$ the distance of the eye. The representation of the base of the first column may now easily be compleated thus ;

The torus being the largest circular member, I find its representation first, through the extremities t and r of its diameter : I draw t C and r C, which are the perpendicular sides of its circumscribing square : then through r (which is that extremity of the diameter nearest to i) I draw r i cutting I C in o. A line drawn through o, parallel to H L, will be a parallel side of a square circumscribing the torus. If these three points t o r, assisted by the sides of the circumscribing square are deemed insufficient to determine the representation ; more may be found at pleasure by prob. 9. § 86. and indeed it will be necessary to determine this with some accuracy, because the curve which represents the extreme *rim* of the torus will much assist us in forming the other circular mouldings of the base.

The extremities of the *profile* of the base form two points, through which the curve representing each particular moulding must pass ; and a third point will be obtained by laying a ruler by i, and the extremity of any moulding whose representation is wanted ; which will cut I C in the third point sought. Thus d and n are two points, through which
the.

the curve, representing the top of the base, must pass; and a ruler laid by i and d cuts the line I C in b. (*Vide* Cor. 2. § 78.) wherefore a regular curve drawn through the three points d, b and n will represent the circular edge of the top of the base; and to draw this fair and even the curve t o r, found before, will greatly assist. The other curve lines are drawn in the same manner. Lastly, the perpendiculars S and S, drawn to touch that ellipsis which represents the bottom of the shaft of the column within the *cincture*, (*Vide* page 13.) will represent the apparent extremities of the shaft of the column.

The point 2, on the line p q was before determined, and if the distance a 2 is repeated on the line p q, the centers of the tops of all the bases will be obtained; (*Vide* § 58) thro' which perpendiculars being drawn, will represent the axes of the columns, from whence their representations may be determined as before.

After having compleated the representation of the first column, several abbreviations will spontaneously offer themselves to the practitioner who understands the theory which has been taught in the foregoing pages. But this will not be the case with him who only learns a few practical rules by rote: he, like the traveller in a strange country dares not deviate from the high road, though ever so round about, whilst the other, like one who

is well acquainted with the ground, takes the shortest and pleasantest, altho' less frequented, paths, and reaches his journey without finding it wearisome or tedious.

E X A M P L E XIII.

Fig. 54 exhibits another representation of the same row of columns as were represented in the last example: but here instead of supposing the picture to be parallel to a line (A n fig. 32) passing through the centers of the columns, as P P, (fig 32) the picture is placed in the most advantageous position (as p p fig 32) between the eye I (fig 32) and the said colonade; according to the directions given in page 88.

I first draw any indefinite line, H L (fig 54) for the *horizontal line*, in which I assume the point S for its center and draw a perpendicular S I, equal to S I in fig. 32; then is the point I (fig. 54) the place of the eye. (*Vide* § 23, page 14).

I next lay, on the *horizontal line*, (H L) from its center (S) S V, equal to S V in fig. 32; then is V. (fig. 54) the *vanishing point* of all lines, parallel to the ground and perpendicular to the row of columns. Also, through I, I v is drawn perpendicular to I V, and cutting H L in v, which is the *vanishing point* of lines parallel to the colonade and to the ground. (*Vide* § 25, page 16). I next lay the distance I v of the *vanishing point*

point *v*, from *v* to *d* on the horizontal line, and the distance (*I V*) of the vanishing point *V*, from *V* to *D* (*Vide* Cor. 2. § 78). Also, at *I*, I make a small square (*Q*) on the distance lines *IV* and *Iv* (*Vide* § 9) and draw its diagonals: produce that which passes through the eye, *I*, till it cuts *HE* in *B*, then will *B* be the vanishing point of one diagonal of all squares whose sides tend to the vanishing points *v* and *V*. Also through *I*, I draw a line *I m*, parallel to the other diagonal of the square (*Q*) but as this will not cut *HE* within the bounds of my plate, I cannot obtain the vanishing point of that diagonal. To remedy this inconvenience, I have recourse to the consideration used in page 122. Through any two points *m* and *n*, in the line *I m*, I draw any two indefinite lines parallel to each other and cutting the horizontal line in *k* and *l* (these lines should be drawn as far apart as possible, and in such a part of the paper as not to interfere with, or confuse, the other parts of the work). I take the distance, *n l* in my compasses, and lay it on that line *n l*, as often as I judge necessary upwards and downwards; (in this example I lay *n l* twice downwards, in the points 4 and 8). Then I take *k m* in my compasses and lay it in like manner on *k m*. I then divide each of these distances, on *k m* and *n l* into the same number of equal parts, (in this example I divide them each into four

four equal parts) and I number these divisions as the figure shews: Then it is evident, from page 122, that any line drawn through two corresponding divisions will tend to the vanishing point of that diagonal of the square, Q, which is parallel to I m. We are now sufficiently furnished with all the points necessary to determine the perspective representation required; the manner of doing which I shall now describe.

1st. On reviewing fig. 32, I observe that the center, (m) of the 1st column is in the continuation of the line I V, which answers to I V in fig. 54; consequently a perpendicular to H L (fig. 54) drawn through V, will be the indefinite representation of the axis of the 1st column.

I next draw E N parallel to H L and at the same distance below it as in the last example; which I assume as the entering line of a plane passing through the tops of the bases, parallel to the ground. From any point, c, in this line (E N) I lay c b equal to V m in fig. 32, then I draw V c and b D, cutting V c in P. (*Vide* §. 78.) Then a line, p q, drawn through p, parallel to H L, will represent an original line parallel to the picture, and passing through the center of the top of the base of the first column; and since this line cuts the representation, (V X) of the axis of the first column in a, that point, (a) is the representation of the center of the top of the base of the first column.

a d and a 2 are obtained exactly are in the last example, (*Vide* page 133.) I take a 2 in my compasses and repeat it on the line p q, cutting it in the points 3, 4 and 5. Then I draw the visual, a v, and cross it by laying a ruler by d and the points 2, 3, 4, 5, and by this means I obtain the points II, III, IV and V, which are the centers of the tops of the bases of the columns.

Esteeming a d as the radius of the lower part of the column, I draw a profile of a Tuscan base, exactly as in the last example. Then I find the representation of the outward rim of the torus by the same means as directed page 134; that is, through t and r I draw (t S and r S) the perpendicular sides of the circumscribing square. Also, I draw a line, o S, through the center S of the horizontal line, and the center of the profile of the torus; a ruler then by r and i cuts this line, o S in o. Through o draw a line parallel to E N, which will be a parallel side of the circumscribing square. Then draw a regular curve through the 3 points t, o, r, or more may be found if these are deemed insufficient. (*Vide* page 134.) The other circular moulding are done in the same manner.

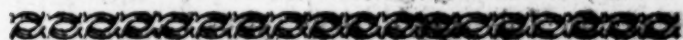
In order to explain the better how to determine the representation of the *plinth*, which is square, I have drawn the *central profile* (which was before supposed determined in its proper place) in the upper part of the plate,

plate, that it may not interfere with any other part of the work: this forms the rectangle $wxyz$. Though f , the center of that face which is visible (which in this instance being above the horizontal line, is the lower face); I draw two lines, 12 and 34 to V and v ; I lay a ruler by d and y , and cut the line 34 in 3 . Also a ruler, by the same point d , and z cuts the said line 34 , in the point 4 ; through these points, 3 and 4 , I draw lines, 35 and 46 tending to V . I then lay my ruler by f and B , and it cuts the lines 35 and 46 in the points 5 and 6 ; through which points, 5 and 6 , I draw lines tending to v , and so complete the representation of the square face of the plinth. It would be needless to say how the representation of the remainder of the plinth is determined, or how to determine the representation of the other columns, after having so minutely described the representation of the first.

In finding all these representations I only make use of the two vanishing points, V and v , the two *dividing centers*, d and i , and the diagonal point, B ; it therefore appears, that there is no occasion for the *dividing center*, D , or to draw any lines tending to the remote diagonal point beyond the plate. (*Vide* page 137.) But certain cases may happen (particularly in the more ornamental orders which have many concentric square mouldings) where

where lines tending to both the diagonal points, will save much trouble, and therefore I was willing to furnish the learner with whatever can be serviceable in any future case. But I must again remind the learner, that his own good sense and knowledge of the Theory of this art, will teach him more practical contractions than can be derived from books; nothing being more true than Mr. Pope's observation,

Good sense, which only is the gift of Heav'n,
And tho' no science, fairly worth the seven.



A digression concerning the curve formed by the perspective representation of an original circle, situated in a plane oblique to the picture: with a specimen of the profound mathematical and optical knowledge of the CRITICAL REVIEWERS.

In the preceding pages I have several times used the word *Ellipsis*, to express the representation of a circle, for which, perhaps, I may incur the same censure from the authors of the Critical Review, which they bestowed on Mr. Ware, in the year 1756. In their critique of that gentleman's translation of Saragatti's Perspective, they tell us,

“ His

“ His regular ellipsis for the representation
 “ of a circle is a notorious mark of his want
 “ of mathematical and optical knowledge.

“ In regard to the regular ellipsis for the
 “ representation of a circle; it appears from
 “ the very nature of perspective, that the
 “ fore part of a circle will *appear* more round
 “ than the back part, which being farther
 “ removed from the eye, cannot appear to
 “ have the same degree of curvature, and
 “ consequently the whole figure, if drawn,
 “ must be very far from having the form of
 “ such an ellipsis as is to be made by a trans-
 “ verse and conjugate diameter.” Vide Crit.
 Rev. for July 1756, page 509.

Now I must beg these *able* criticks to re-
 collect (for nobody can suppose any mistake
 of theirs to proceed from ignorance) that the
 rays which come to a spectator's eye, from
 every point in the circumference of a circle
 which it views, form a cone: and since the
 section of these rays by the plane of the
 picture, is the *perspective representation* of that
 circle. (*Vide* page 2.) It follows, that the
perspective representation of a circle on a
 plane, must always be one of the conic-sec-
 tions. If they read over the definitions
 which *Apollonius Pergæus*, * and after him all
 the writers on the conic-sections have given
 of

* Vide Dr. Halley's edition of Apollonius, fol. Oxon
 1710; page 13, &c.

of the PARABOLA, HYPERBOLA & ELIPSIS (except Descartes and some other writers on curves in general, who define them by their properties, when described on a plane) and if they can conceive themselves situated near the glass door of a circular building, as a summer-house, &c. then I think it will occur to them, that the circular boundaries of the ceiling and floor (or so much of them as are visible) will be represented on the glass by two ellipses, or elliptical segments *—if they conceive the door to be removed within the building and themselves advanced in its place, (so that the eye may be in the circumference of the building) the visible parts of the said circular boundaries will form on the glass two *parabola*: and these will become hyperbolæ, if they conceive their eye advanced within the building and nearer the glass. If they cannot conceive all this, I would advise them to aid their fancy by a trip, some summer evening, to a new round building near Cold-Bath-Fields, called the *Pantheon*; or, if more convenient, to Ranelagh; where, when they have considered the above phenomena, I would have them sit down and endeavour

* Except in one particular case only, which is, when the cone, formed by the rays, which enter the eye from the said circular boundaries, is cut by the door in a position sub-contrary to its base; in which case the representation will be circular. Vide Apollonii Pergæi lib. 1. prop. 5.

deavour to enter a little into the urbanity and good humour of the company, which, perhaps, will unbend their austerity, and send them home in a temper of mind less liable to subject them to contempt for censures either too harsh, or totally unjust, as in the present case.

Nevertheless, the reader must be informed, that however excellent the conic sections are in many branches of the mathematics, we can derive nothing from them to facilitate the *practice of perspective*. It would require a good deal of theory to explain their organical construction, and still more time and study to adjust them to perspective representations of the circle; and after all, the method would be much more operose and troublesome than the general one taught in the last problem, for which reasons the theory of conic sections, and their application to perspective, are omitted in this Work. But those who wish to see the circle represented on a plane, by considerations drawn from the conic sections, may consult Mr. Hamilton, who has learnedly and copiously handled that subject, in the Third Book of his Body of Perspective.

It may also be necessary to acquaint such of my readers as do not think themselves so profound as the reviewers, and who may be misled by the seeming plausibility of the argument used by those learned
criticks

criticks, (" that the fore part of a circle
 " must *appear* more *round* than the back
 " part") that the curve $A B C D$ (fig. 21.
 page 59.) which is a just *perspective* repre-
 sentation of a circle, is an ellipsis; whereof
 $A B$ is the *transverse* or *longest* axis; $C D$
 is the *shortest* or *conjugate* axis; the points,
 F and F are the *foci*; and c , is the cen-
 ter. Also the curve, $D n F f$, (fig. 52)
 which is the *representation* of a circle, as
 found by problem 9; is likewise a regular
 ellipsis; of which $a B$ and $M N$ are the *trans-*
verse and *conjugate* diameters; c is the center,
 and F and F are the foci; but it is certain
 that in both these examples the point C ,
 which is the *representation* of the center of
 the *original* circle, does not coincide with the
 center of the ellipsis; and therefore the line
 $8, 9$, in fig. 21, or the line $D F$, in fig. 52,
 which are the representations of a diameter of
 the original circle, is not a diameter of the el-
 lipsis which represents that circle; conse-
 quently the segment $8 A 9$ (of the ellipsis, fig.
 21.) or the segment, $D F f$ (of the ellipsis,
 fig. 51.) which are the representations of the
 most remote *original* semi-circles, are less than
 the segment $8 B 9$ (of the ellipsis, fig. 21.) or
 the segment, $D n F$ (of the ellipsis, fig. 51.)
 which are the representations of the *original*
semi-circles, nearest the picture; and this is
 what I apprehend has led our criticks into
 their too confident assertion.

L

A di-

A digression concerning the representation of a row of equi-distant cylinders.

If we inspect the *representation* of the 3 cylinders (fig. 48.) we shall perceive that the *representation* of the cylinder marked III is wider than that marked II, and this is again wider than that marked I, and if the number of these cylinders was increased the *perspective representations* of those near the center of the picture would be less than of those more remote.—It is very strange that dispute should find entrance in a subject which may be brought to the test of *Geometrical reasoning*, and the truth or falshood of every assertion incontrovertibly decided. Yet we are told by a gentleman, whose writings on this subject, together with his patronage by the great, intitle him to some attention, that this manner of representing a row of cylinders “ which has long subsisted, though he apprehends, founded upon erroneous principles, *has occasioned much controversy among painters, mathematicians and other ingenious men.*” If these ingenious men were of the same opinion with this ingenious writer, I wonder not that the mathematicians dissented from them; but I wonder they should dispute *much* about it, because they might be certain their opponents were ignorant of those few *geometrical præcognitæ* which alone could render them capable of conviction: nor do I now undertake to consider this gentleman’s

tleman's arguments with any hope or desire of lessening their rectitude in his opinion, but purely to hinder the learner from conceiving any imperfection in this art ; which being established on *geometrical reasoning*, will never require the sense of *seeing* (which our author calls the most *infallible guide* on this occasion) to correct it, but only to applaud and admire its agreement with nature.—I should be very glad to avoid personality if it were possible, but notwithstanding, this gentleman tells us "*his opinion is neither new nor singular*," I cannot consider it as very old or general ; because I never saw any book printed before his wherein any such disputes or opinion are mentioned : and indeed the whole would be unworthy of notice were it not that the reputation of this gentleman (which excepting in this instance is not wholly undeserved) may give a sanction to errors tending to lessen the utility of perspective, which if it required correction by the dictates of fancy (which every man has a right to consider as the result of his own *observation and experience*) it would be a vague and uncertain art, scarcely worth studying.

In order to avoid any misrepresentation, I shall set down the gentleman's own words, and placing my remarks opposite to them, leave my reader to adopt that opinion which shall appear to him best supported by reason and evidence.

His words are,

“ Suppose it was
 “ required to draw
 “ the representation
 “ of a range of co-
 “ lumns parallel to
 “ the plane of the
 “ picture; if they are
 “ drawn according to
 “ the strict *Rules of*
 “ *Perspective*, then
 “ that column which
 “ is in the center of
 “ the picture will be
 “ the least, and con-
 “ frequently, those on
 “ each side of it will
 “ be larger and larger
 “ continually the far-
 “ ther they are re-
 “ moved from the
 “ center of the pic-
 “ ture: But to ex-
 “ plain this more ful-
 “ ly, let K L M N,
 “ (*Vide* fig. 32.) be a
 “ plane which passes
 “ through the eye pa-
 “ rallel to the ground,
 “ then will P P be
 “ the horizontal line,
 “ and c the center of
 “ the

Dr. Brook Taylor in his
 Book printed 1719, page
 1, says,

“ To have a complet
 “ and clear notion of the
 “ principles of this art, let
 “ the reader consider that
 “ a picture drawn in the
 “ utmost degree of perfec-
 “ tion, and placed in a
 “ proper position, ought
 “ so to appear to the spec-
 “ tator that he shall not
 “ be able to distinguish
 “ what is there represented
 “ from the original objects
 “ actually placed where
 “ they are represented to
 “ be. In order to produce
 “ this effect, it is necessa-
 “ ry that the rays of light
 “ ought to come from the
 “ several parts of the pic-
 “ ture to the spectator's
 “ eye, with all the same
 “ circumstances of direc-
 “ tion, strength of light
 “ and shadow, and colour,
 “ as they would do from
 “ the corresponding parts
 “ of the real objects seen
 “ in their proper places.—
 “ Wherefore in the de-
 “ monstrations of the fol-
 “ lowing propositions in
 “ this book, we must al-
 “ ways have recourse to
 “ this *General Foundation*,
 “ by shewing that the rays
 “ of light will come in
 “ the same directions from
 “ the

" the picture, let A B
 " and g h be *two* *
 " columns cut by this
 " plane, then let lines
 " be drawn from the
 " extremities of these
 " circles to the eye
 " (I) and the sections
 " a b and c d, with
 " the picture P P are
 " the *projections* of
 " those circles upon it ;
 " and by measuring
 " these representati-
 " ons we shall find
 " that c d is much
 " longer than a b.—
 " from whence we
 " see that the farther
 " any column is re-
 " moved from the
 " center of the pic-
 " ture, the larger will
 " be its representati-
 " on. Now the ques-
 " tion is, whether co-
 " lumns situated in
 " this

" the several parts assigned
 " in the picture, as they
 " would do if they came
 " from the corresponding
 " parts of the original ob-
 " jects placed in their
 " proper situations."

Now I suppose it will
 not be denied that the rays
 which proceed from the
 projections a b and c d are
 the very same as proceed
 from the original circles G
 and H to the eye, (I)
 and therefore they must
 enter the eye in all the
 same circumstances of di-
 rection, as those rays which
 come from the original cir-
 cles, G and H ; conse-
 quently if a b and c d are
 not the true *perspective re-*
presentations of the circles
 G and H, it follows, that
 Dr. Taylor's GENERAL
 FOUNDATION is erroneous,
 and the whole of his rea-
 soning and conclusions
 built upon it, must be
 false.

Wherefore the sum and
 substance of what this ge-
 ntleman seems going about,
 is to prove that Dr. Brook
 Taylor's

* I have varied my scheme a little from his, having put
 a range of five columns extending from the center of the
 picture one way only, being more expressive of what I
 shall want to point out hereafter. But this can oc-
 casion no difference, as the first column and that which
 is most remote will give a just idea of what he is saying.

“ this manner are to
 “ be thus represented
 “ on the picture or
 “ not.”

Taylor's principles of perspective, on which he has bestowed such encomiums, and whose truth he has laboured to establish both by his ingenious schemes and argumentations, are, nevertheless, void of truth, and that he has only been amusing his readers with a false light which he is now going to extinguish, and leave them in the same darkness in which he found them.

The only want of correctness observable in Dr. Brook Taylor is in the Title page, and the first definition of his book printed in 1715. His Title page is “ LINEAR PER-

“ The definition
 “ we have given of
 “ the word perspec-
 “ tive is this, viz. *to*
 “ *draw the representa-*
 “ *tion of objects as they*
 “ *appear to the eye.*
 “ And we have avoid-
 “ ed the more general
 “ definition, viz. of
 “ drawing the repre-
 “ sentation of objects
 “ by the *Rules of*
 “ *Geometry, &c.* as
 “ the former appear-
 “ ed more significant
 “ of

“ SPECTIVE : or a new
 “ method of representing,
 “ justly, all manner of
 “ objects *as they appear to*
 “ *the eye,* in all situati-
 “ ons.” And when he
 defines Perspective, in his
 first definition, he says,
 “ Perspective is the art of
 “ drawing on a plane the
 “ *appearances* of any figures
 “ by the rules of geome-
 “ try.”

The Doctor's Title page and definition put together amount to this, viz. Perspective is the art of representing objects *as they appear to the eye,* by the rules of geometry. But this

“ of what we intend-
 “ ed to exprefs by
 “ the Term Perspec-
 “ tive.”

this definition, although far from a good one, is not at all mended by what this gentleman adopts. His only objection lies againſt the *rules of geometry*, and he retains the impropriety that perspective is the art of drawing the representations of objects as *they appear to the eye*, which I ſhall endeavour to ſhew is not the buſineſs of perspective.

It muſt be remembered that the *apparent* length of a line does not depend on its real length, but on the angle which it ſubtends at the eye: in proportion to which its appearance (*Vide* page 78 and 79) will occupy a greater or leſs ſpace on the retina. (Thus becauſe the line A B, fig. 30. ſubtends a larger angle, A P B, at the eye than the line E F does; its *appearance* (a b) occupies a ſpace on the retina larger than c f, which is the *appearance* of E F).

Wherefore all lines that are ſeen under the ſame, or under equal angles, appear equal altho’ ever ſo different in length and poſition. Thus it is evident that c d (fig. 32.) appears equal to g h, although it is much larger and ſituated nearer to the eye, (at I) becauſe the angle d I c, which it forms at the eye

is the same as the angle $h I g$, which the line $g h$ forms at the eye.

It will be no difficult matter now to prove that the width of the column G (fig. 32.) is seen under a larger angle ($A I B$) than the width of the column H , which is seen under the angle $g I h$ or $c I d$. For from the centers m and n , of the circles G and H (fig. 32.) draw $m A$, $m B$; and $n g$, $n h$; which will be perpendicular to the tangents $I A$, $I B$, $I g$ and $I h$, (by E. 3 18.) also $I n$, is longer than $I m$, (by E. 1. 32, Cor. 3 and E. 1. 19) wherefore $g I$, and $h I$ are longer than $I A$ and $I B$. On these lines lay from the points g and h , (by E. 1 3.) $g i = I A$ and $h k = I B$ then in the right angled triangles $I m A$ and $i g n$ since two sides $I m$ and $m A$ and the included right angle in the one ($I m A$) are equal to two sides $i g$ and $g n$ and the included right angle in the other ($i g n$) it follows (by E. 1. 4.) that the angle $g i n = A I m$: in the same manner it may be proved that the angle $h k n$ is $= m I B$. But the angle $g i n$, is greater than $g I n$ (by E. 1 16.) also the angle $h k n$

is greater than hIn ,
whence their sum $gin +$
 $hkn (= AIB)$ is greater
than $gIn + hIn (= g$
 Ih or $cId)$

Since then the circle H
is seen under a less angle
than the circle G ; if ob-
jects are to be represented
as *they appear to the eye*, the
circle H ought to be repre-
sented by a less line than
the circle G is repre-
sented by: that is, cd ,
which is the *representation*
of H , ought to be less than
 ab which is the *representa-*
tion of G : but cd is much
longer than ab , yet its
appearance occupies the
same space on the retina of
an eye fixed at I , as the
appearance of the origi-
nal circle H does; where-
fore if in conformity to
this rule of "representing
objects as *they appear to*
the eye," we make cd less
than ab its *appearance*
will not occupy the same
space on the retina of the
eye at I , which the origi-
nal circle H , does; conse-
quently, when objects are
represented on a plane (P
 P) as *they appear to the eye*,
(at I) such a representation
will not afford the same
appearance as the original
objects themselves.

It is the business of per-
spective to deceive the eye:
and

and this can never be done so well as when the *picture formed on the retina* (*Vide page 78 and 79.*) by viewing the *representation*, is the same as the *picture formed on the retina* by viewing the original objects, but this I have shewn cannot be the case if the representation of the columns is drawn on the picture P P similar to their *appearance on the retina* of the eye at I. Consequently it is not the business of perspective to draw the *representations* of objects as they *appear to the eye*: but its business is to teach us to draw such a representation of objects, on any plane (P P) fixed in any assigned situation respecting the eye (I) and the said objects, as shall strike the eye in the very same manner as the real objects themselves do. And this Dr. Taylor particularly enforces as the *foundation of the whole art*, in the words I have before quoted from him (page 148).

In his second treatise published in 1719 he seems well convinced of the improprieties I have noticed in his first treatise. The title being, "New principles
" of Linear Perspective;
" or the Art of designing
" on a Plane the REPRESENTATIONS

“SENTATIONS of all sorts
“of objects.” And his
first definition is expressed
in these words, viz. “Li-
“near Perspective is the
“art of describing exact-
“ly, on a plane surface,
“the REPRESENTATION
“of any given object.”
In both these places the
former impropriety is re-
moved by substituting the
word “*Representation*.”
instead of “*Appearance*.”
For the different meanings
of these words, see page
78 and 79.

“For since the
“fallacies of vision
“are so many and
“great and since we
“form our common
“judgment and esti-
“mate the ^b appear-
“ance of objects
“from custom and
“experience and not
“from mathematical
“reasoning: there-
“fore it seems rea-
“sonable not to
“comply with the
“strict rules of ma-
“thematical per-
“spective in some
“particular cases (as
“in

^a Were not the falla-
cies of vision many and
great there could be no
such arts as perspective or
painting whose only busi-
ness is to deceive the eye;
which, were it infallible,
would never take a plane
surface, however it might
be covered with lines, and
colours, for a variety of
objects of which some
seem near and others afar
off. The fallacy of vision
then seems to be the only
true assertion in this pa-
ragraph: and because it
is a truth, our author
(notwithstanding he sup-
ports it with 2 long quo-
tations from Dr. Smith,
and Mr. Locke) denies it
himself in the preface to a
more magnificent work of
his

“ in this before us)
 “ *but to draw the*
 “ *representations of*
 “ *objects as they ap-*
 “ *pear to the eye.*”

his, and supports this favourite opinion by telling us that the eye is the most infallible guide.

^b We do not estimate the “ appearance of objects “ by custom.” The appearance of objects is formed on the retina, (*Vide* page 78 and 79) and therefore is not a subject of estimation : but from the appearance of objects we judge of their distance, magnitude and positions, by the help of custom and experience.

^c Since this gentleman is so desirous to have objects represented as they appear to the eye, let us enquire how this row of columns appears to the eye ; and we shall find by drawing lines from the eye, at I, (fig. 32.) to touch the circles which are the sections of the columns by the plane, K L M N, that the ^a intercolumnations do actually appear less and less ; so much so, that the column marked III, appears almost close to that marked IV, and a part of the column H is hid by the fourth column (*Vide* fig. 53 ; where the perspective representation of the lower part of a row of columns, is drawn by the *strict Mathematical*

“ For if the above columns are
 “ represented according to the strict
 “ rules of this art ;
 “ then the columns,
 “ as they recede from
 “ the center of the
 “ picture, will be
 “ thick and clumsy,
 “ their

" their intercolumna-
 " tions will be less
 " and less,

" " and the whole
 " beauty of the build-
 " ing will be entirely
 " destroyed."

thematical Rules explained in page 132, &c. in which these particulars respecting the *appearance*, of the columns are preserved) consequently, if we would draw the representation of the row of columns as *they appear to an eye* placed opposite to one end of the colonade, at a short distance from it, the intercolumnations must be made less and less till the columns touch each other; and if they are still continued, the representation of that which is nearer should hide a part of the next which is more remote.

* If the eye is placed at I, (fig. 32.) the spectator will immediately reflect that he is not in a situation to judge of the equality of the intercolumnations, nor of the beauty of the building: in order to get a better sight of these he will naturally retreat farther off, and whilst he is doing this, he will perceive the intercolumnations to open themselves, so that when he is at the distance of ten times the length of the whole range of columns, in the direction of the line I m, the columns will *appear* very nearly of an equal width, and their intercolumnations will also *appear* very

very nearly equal. And if they are represented on the plane P P according to what our author calls the *strict Mathematical Rules of Perspective*, the representations of their *apparent widths* will have no discernible difference, and the representations of the intercolumnations will measure very nearly the same; all which is evident from the scheme (fig. 32.) where the dotted lines, which touch the circles (and tend to a point in the line I m produced, equal to ten times the length of the whole range of columns, from the point m,) mark on the line P P the representations of the widths of the columns and their intercolumnations.

From these observations we may conclude, that when the eye is supposed in such a situation, with respect to the original object (as in the point I, fig. 32) that the beauty and just disposition of its parts cannot be seen; in such a case the *true Perspective Representation* of that object will conceal that beauty and disposition. And that this same *Mathematical Perspective* (always true and consistent) reveals all the beauty, and shews the disposition

tion of the parts of any object, if the eye is supposed situated where they may be seen to advantage.

But although the line c d (fig. 32.) appears less than the line a b to an eye which views them from I: and the *representation* of the column H (fig. 53.) appears less than the representation of the column G, when viewed at the distance I C; yet at any other distance the said *representation* appears clumsy; but this is not owing to the fault of perspective, but to the bad position of the picture P P (fig. 32.) which although parallel to the range of columns, is very oblique to the eye; for when the eye views this range of columns, the retina will be naturally placed parallel to the line p p (*Vide* page 88) which on that account is the proper position for the picture: and if the picture is supposed in this position, and the representation is drawn according to the *strict Mathematical Rules of Perspective*, as is done in fig. 54. (*Vide* Exam. 13. page 136) no such clumsiness, as this gentleman complains of, will be observable; but the column G, which appears largest, has the largest representation.

The

" What has been
 " said upon this sub-
 " ject, relates princi-
 " pally to round or
 " cylindrical bodies,
 " such as globes, co-
 " lumns, or the like;
 " but as to angu-
 " lar ones (especial-
 " ly those that are
 " square) since their
 " apparent widths
 " are perpetually in-
 " creased the more
 " obliquely they are
 " seen by the eye,
 " therefore, the re-
 " presentations should
 " grow longer and
 " longer in width the
 " more they are re-
 " moved from the
 " picture."

The frontispiece to this gentleman's work exhibits the absurdities which may be committed by a person who makes a design without some knowledge of perspective, which is indeed a tacit encomium on the art worthy of its contriver the ingenious Mr. Hogarth. I should imagine that our author, in what follows, means a similar compliment to the *Science of Geometry*, by bringing together, in this place, all the absurdities which a man may commit who attempts to reason on a mathematical subject, without some knowledge of its elements.

I suppose by *Square Objects* this gentleman means Prisms standing on square bases, the apparent magnitudes of which do *not* grow larger and larger *perpetually*. To demonstrate which suppose f u (fig. 32.) to be a tangent line, touching in the point f , a circle whose center is I . It is then evident from inspection, that the divisions of this tangent line increase continually from f towards u , and that any equal number of divisions always subtends the same angle at I : thus the space of the tangent line between 60 and 65 subtends an angle
 at

at I, of 5° as well as the space between f and g .— This being allowed, let us suppose the square Q (fig. 32.) to be carried towards R , between the two parallel lines, rt and fu ; then it is evident, that if the square Q , was every where to have the same *apparent magnitude*, the space, which it seems to occupy on the tangent line, fu , ought to increase exactly as the tangents increase, (for then it would always be seen under the same angle) but, the increase of the divisions of the tangent line is at first very little, but as they advance towards u , the divisions grow very large; whereas the space which the square Q , will seem (to the eye, at I) to occupy on the line fu , will increase uniformly; because the point f will appear to move uniformly faster than r in the ratio of rI to fI ; so long therefore as the increase of the space which the square Q , seems to occupy on the tangent line fu , is greater than the increase of the tangents, the *apparent width* of the prism will increase; but as soon as the tangents increase faster than this space increases, then the *apparent width* will begin

M to

to *decrease*; and continue so to do till, when it has got at a great distance, it will be too small for perception; for although the space it *seems to occupy on the tangent line* may be very long; yet that *space* may only be a small part of one division on the tangent line, in which case the *apparent width* of the prism will be only a small part of a degree.

b " Thus the representation of the square, Q, (*Vide* fig. 32) which is seen only in front, cannot appear so long as the representation of the square, R, which is viewed anglewise."

c " We say that the *apparent magnitude* of objects that are square or triangular will be greater when viewed anglewise than when seen in Front."

b In the scheme which the gentleman refers to, the *apparent magnitude* of the prism R (*Vide* fig. 32) is in its *increasing state*, and therefore subtends a greater angle at the eye than Q, in my scheme (fig. 32) R is in its *decreasing state*; and subtends a less angle at the eye, which makes both it, and its *representation* (u x) appear smaller than Q.

c I have shewn that square prisms do not always appear greatest when seen anglewise; and triangular prisms whose bases are *equilateral* will ever appear *smallest* when seen anglewise; for the *apparent breadth* of a triangular prism will always be bounded by one of its sides, which must appear largest when placed immediately next the eye.

b This

d " But the *appa-*
 " *rent magnitude* of
 " columns, or any
 " other round objects
 " *will always be the*
 " *same at the same*
 " *distance :*

" Because in the
 " first case, the dia-
 " gonal of a square
 " (*e* which in some
 " views measures its
 " apparent width) is
 " larger than its sides;

" But in the lat-
 " ter case the dia-
 " meter of a circle
 " (*f* which constantly
 " measures its *appa-*
 " *rent width*) is al-
 " ways of the same
 " length."

" And therefore to
 " represent columns
 " larger

d This is very true, but
 has nothing to do with a
 range of columns, which
 surely cannot all be equally
 distant from the eye.

e Here a truth is advan-
 ced (tho' by no means con-
 clusive) to prove a falshood,
 as I have shewn page 161
 and 162.

f Here a falshood is ad-
 vanced to prove a truth.
 for the diameter of a circle
never measures its appa-
 rent width. The apparent
 width of any circle (as H,
 fig 32) is measured by the
 angle g l h, formed by
 two tangents touching that
 circle and meeting in the
 eye; (I) but if those tan-
 gents (g I and h I) touched
 the circle at the extremity
 of a diameter, they would
 be perpendicular to it (by
 E. 3. 18.) and therefore
 would never meet in the
 eye (at I by E. 1. 28.)

g This conclusion is some-
 what more dogmatical than
 his other assertions, but e-
 qually void of truth; for I
 have before demonstrated
 that although c d (fig. 32)
 which is the representation

M 2

of

“larger and larger,
 “when they are at a
 “greater and greater
 “distance is, we pre-
 “sume, false in the-
 “ory, (in an optical
 “sense only) and
 “cannot be true in
 “practice.”

of the circle H, is larger
 than a b, which is the re-
 presentation of the circle
 G; yet the said e d, ap-
 pears to the eye, at (I)
 less than a b, and therefore
 such a representation cannot
 be false in an optical sense:
 and even if we would re-
 present objects as they ap-
 pear to the eye, (which this
 author so warmly recom-
 mends) then it follows
 that if any object is so
 posited as to hide part of
 another object; the repre-
 sentation of that object
 should hide a part of the
 representation of the other:
 thus because the fourth co-
 lumn (fig. 32) hides a part
 of the column H, (from
 the eye at I) its represen-
 tation should hide a part of
 the representation of the
 column H: which is the
 case when those repre-
 sentations are drawn by
 the rules of *strict Mathematical Perspective*, as is evi-
 dent from fig. 53 and 54
 before-mentioned; so that
 we may infer that this same
Mathematical Perspective is
 neither false in theory nor
 in practice.

The gentleman in his
 work now before me con-
 tents himself with telling
 us that *strict Mathematical Perspective is false in theory and practice*, and does not
 inform us of any method
 more

more certain or true : but in his magnificent work, published at the expence of royal munificence, he is so obliging as to let us into this secret, which is by drawing the *representation* of the columns always of the same width, and to make the representation of the intercolumnations equal : but if the column H (fig. 32) is hid in part by the column IV, it is evident, that if the representations of these columns are drawn as wide apart from each other as the representations of the columns G and H, they will not be represented as they appear to the eye (at I) and therefore such a representation is false, even when measured by his own *Rules of Restituta*.

It would be tedious both to myself and my reader to continue the examination of this chapter any further ; which contains several absurdities equal to any I have pointed out. The whole is a lesson of the usefulness of those few geometrical elements I have prefixed to this work, which will not only restrain the student from broaching erroneous notions himself, but also hinder him from being imposed on by those of others.

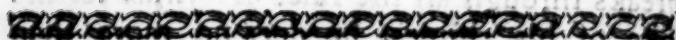
I would

I would not by any means have it understood that all this gentleman's works are as exceptionable as those parts I have noticed, on the contrary they contain better practical rules than I have seen in any book published before his; nor have I said any thing with an intention to depreciate him, but purely to vindicate this art from false aspersions. I agree with him that every little minutiae in a picture need not be adjusted by the strict rules of this art, nor would I recommend it to a portrait painter to measure the prominences and recesses of the face and then attempt to draw its likeness by the rules of perspective: the good sense of the student will point out to him that when he has adjusted the capital parts of his piece by these rules, the less and inferior parts may be supplied by the hand alone; but in this case it is the business of the hand to follow *the rules of art* and not to mend them. He will always find that if he has chosen a proper position for the eye and the object, the more he varies from perspective precision the less perfect his piece will be; and too much licentiousness will subject him to the imputation of ignorance, more than even a nice compliance with the rules of perspective will lessen the freedom and beauty of his performances.

The poet has as much right to complain of the fetters of grammar as the artist has
of

of those of perspective, when the former breaks through all the rules of syntax for the sake of his rhimes, he does no more than he who rejects all the rules of perspective under pretence of its cramping the grace and freedom of his piece; and as we admire the poet who blends harmony and correctness together, so the artist is most commendable who unites *Beauty* and *Grace* with *Propriety*; and whose works proclaim him, at once, a Master of Art, and a Disciple of Science.

C H A P.



C H A P. VIII.

The theory of vanishing lines, with its application to the inclined picture; and to objects situated on planes not parallel to the horizon, or standing on a hill ascending or descending from the picture.

SINCE every original plane must be either parallel or inclined to the picture, the learner has been fully instructed to find the representation of plane objects in all situations: but it was always supposed when the original plane was oblique to the picture, that its *vanishing line* with its *center and distance* was given or assumed. It remains now to shew how the *vanishing line*, with its *center and distance*, of any such plane may be found, when we know its true situation either with respect to the picture, or to some other *original plane* whose *vanishing line* and its *center and distance* is already determined.

T H E O R E M XVIII.

LXXXVII. *The vanishing points (D and d, fig. 4.) of all original lines (AB and BC) situated*

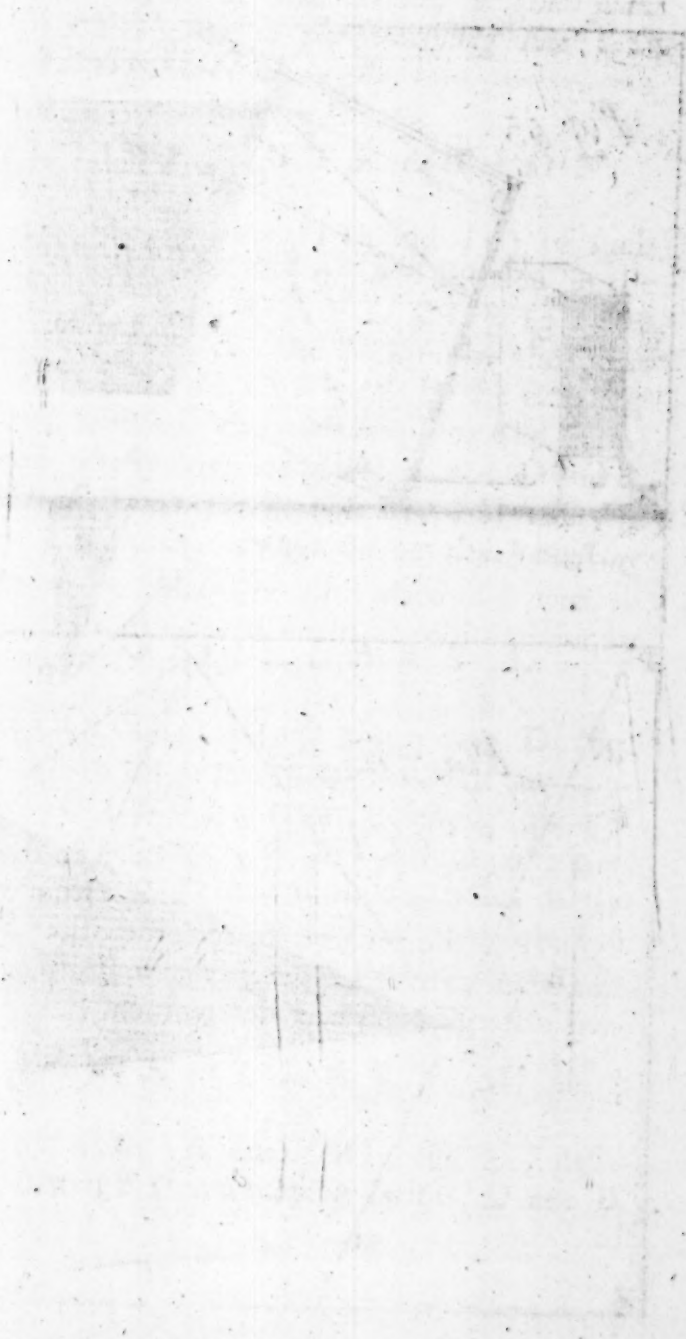


Fig. 55.

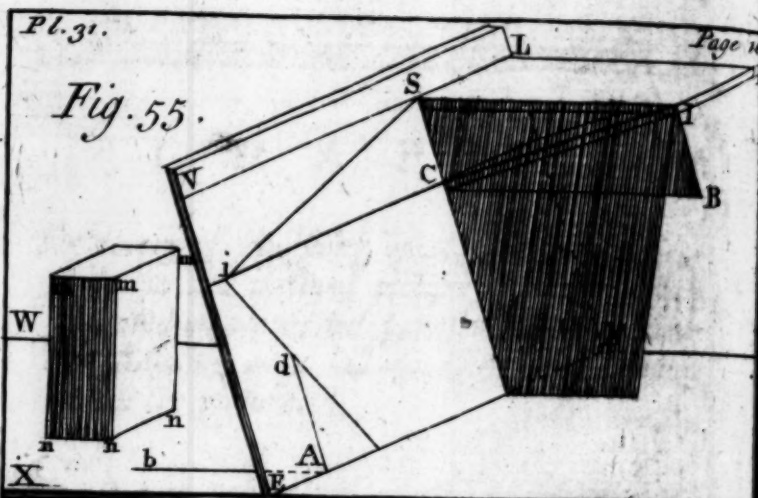
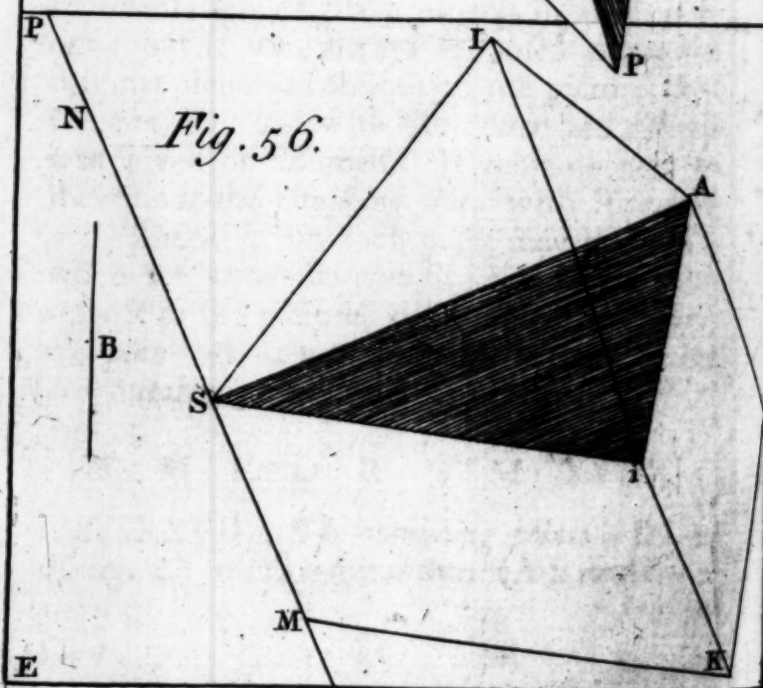


Fig. 56.



situated in, or parallel to, any original plane (WXYZ) are in the vanishing line VL, of that original plane.

DEMONSTRATION.

For the *distances* (ID and Id) of such original lines (*Vide* § 9.) are situated in the vanishing plane, OR VL (by § 13.) and therefore must cut the picture in the same line, VL, in which it is cut by the *vanishing plane*, which is the *vanishing line* (by § 3.) consequently the *vanishing points*, D and d, being sections of the *distances* ID and Id (by § 9) must always be in the *vanishing line*, VL. QED.

COROLLARY I.

LXXXVIII. Hence the point D (fig. 4.) where the representation a b of any *original line* (AB) cuts the *vanishing line* (VL) of the plane (WXYZ) in which it is situated, or to which it is parallel, is the *vanishing point* of the said original line (AB) and of all other lines parallel to it, (by § 17.)

COROLLARY II.

LXXXIX. A line (VL, fig. 4.) passing through the *vanishing points* D and d,
of

of any two original lines (AB and BC) is the vanishing line of the original plane ($WXYZ$) in which those lines are situated.

DEFINITION I.

XCI. Let $IRSL$ (fig. 55.) be the vanishing plane of any original plane $WXYZ$ (Vide § 3.) intersecting the picture in the vanishing line VL , whose center is S , and distance SI (Vide § 7.) I , being the point of sight or place of the eye (Vide § 1.) Then a line (SC) drawn on the picture through the center (S) of any vanishing line, perpendicular to it, is called the VERTICAL OF THAT VANISHING LINE.

DEFINITION II.

XCI. The vertical plane of a vanishing line, is a plane ($SCIB$) passing by its distance (SI) and vertical (SC)

THEOREM XIX.

XCI. The angle (CSI) formed by the distance (IS , fig. 55.) and vertical (SC) of any vanishing line, (VL) measures the inclination of the vanishing plane ($IRSL$) and the picture ($VLEN$) to each other ; which is the same as the inclination (bAd) of the original plane ($WXYZ$) and the picture.

DEMONSTRA-

DEMONSTRATION.

For the *distance* IS, and the *vertical* SC, both cut the *vanishing line* VL, at right angles in the point S, and therefore measure the inclination of the planes VLEN and IRS L. (by E. 11. Def. 6.) Again, through any point A, in the *entering line* (EN) draw Ab in the *original plane*, (WXYZ) and Ad on the *picture*, each perpendicular to EN, then does the angle bAd measure the inclination of the original plane and the picture, (by E. 11. Def. 6.) but the angle bAd is equal to CSI (by E. 11. 10.) seeing the lines forming these angles are respectively parallel to each other. QED.

DEFINITION III.

XCIII. The point (C, fig. 55.) where the picture is cut by a perpendicular (CI) drawn to it from the eye, (I) is called the center of the picture, and that perpendicular (CI) is called the distance of the picture.

THEOREM XX.

XCIV. Any vertical plane (SIBC, fig. 55.) is perpendicular to the picture; and also to its vanishing plane, IRS L.

DEMON-

DEMONSTRATION.

For the *vanishing line* VL, being perpendicular both to SI and SC, (by § 7 and § 90) is perpendicular to the *vertical plane*, SCIB extended by these lines (by § 91, and E. 11. 4.) wherefore all planes which pass by the *vanishing line* VL, must be perpendicular to the *vertical plane*, SCIB (by E. 11. 18.) but the *picture* VLEN, and the *vanishing plane* IRSI, always pass by the *vanishing line* because it is their intersection (by § 3, page 2.) wherefore each of those planes is perpendicular to the *vertical plane*, SCIB. Q.E.D.

COROLLARY I.

XCV. Hence because all *vertical planes* pass through the eye, (by § 91.) and are perpendicular to the *picture*, their *common section* must be a line passing through the eye perpendicular to the picture (by E. 11. 19.) consequently the *distance* IC of the *picture* is the common section of all *vertical planes* (by § 93.)

COROLLARY II.

XCVI. Hence the *vertical*, SC, of any *vanishing line*, is a perpendicular to it passing through the center of the picture: for (by § 92.) it is a perpendicular passing through

through the center of the vanishing line: and it coincides with the vertical plane (by § 91.) but all vertical planes pass through the center of the picture (by § 95.) wherefore all verticals pass through the center of the picture.

COROLLARY III.

XCVII. Hence a perpendicular (CS) drawn (by E. 1. 12.) from the center of the picture to any vanishing line (VL) intersects it in his center S, and is the vertical of that vanishing line (by § 90.)

COROLLARY IV.

XCVIII. The distance (IS, fig. 55.) of any vanishing line, its vertical (SC) and the distance (IC) of the picture, form on the vertical plane (SCBI) a right angled triangle (SCI) of which the distance of the vanishing line, (IS) is the hypotenuse, and the vertical (SC) and distance of the picture (IC) are the two sides: consequently any two of these being given the third may be found: or if one of them, with either of the angles are given the other two may be found.

PROBLEM X.

XCIX. The vanishing line (VL of any original plane (WXYZ fig. 55.) its center (S) and distance SI, together with the angle b A d
or

or CSI (*Vide* § 92.) which its original plane ($WXYZ$) makes with the picture, being given; to find the center (C) and distance (CI) of the picture; and the vanishing point (P) of all original lines, (mn , mn , &c.) that are perpendicular to the said original plane, ($WXYZ$.)

1st. Through S draw on the picture plane $ENVL$, the vertical SC , perpendicular to the given vanishing line VL , (*Vide* § 90) also draw Si on the picture, making with the said vertical SC , the angle CSi , equal to the given inclination bAd of the original plane with the picture; and make Si , equal to the proposed distance SI of the given vanishing line VL . Lastly, through i , draw iC parallel to VL , and cutting the vertical CS , in C ; then is C the center of the picture, and iC , its distance sought.

It is evident that the operations thus performed on the plane of the picture must give the same center C , and distance CI , as if they had been performed on the vertical plane $SCBI$, (*Vide* § 98.) the demonstration being the same as used in § 27 and 28.

The second part of our problem is now easily performed; for through i draw iP on the plane of the picture perpendicular to iS , and cutting the vertical SC , produced, in P ; then is P the vanishing point, and Pi the distance, of all lines mn , which are perpendicular

pendicular to the given *original plane*, W X Y Z.

DEMONSTRATION.

To understand the reason of this proceeding, let us suppose the *vanishing and vertical planes* S I R L and S C B I to be actually raised in their just situations respecting the picture V L E N; then since the *original lines* m n, m n, &c. are perpendicular to the *original plane* W X Y Z, their *distance* I P, (*Vide* § 9) must be perpendicular to the same plane (by E. 11. 8.) and consequently to the *vanishing plane* S I R L (by E. 11. 14.) wherefore it must be perpendicular to S I (by E. 11. def. 3.) and must coincide with the *vertical plane* (by E. 11. 18.) and intersect the picture somewhere in the vertical S C produced, (by § 91.) as in P, in such a manner as to form on the vertical plane a right angled triangle S I P, whose base is the *distance* (S I) of the *original plane* (W X Y Z) and the angle I S P equal to its inclination (*Vide* § 92.) but the triangle S I P is equal in all respects to S i P by construction, (and E. 1. 26.) wherefore i P will cut the *vertical* C S, in the same vanishing point P, in which it is cut by I P; and i P, is equal to the distance I P.

COROLLARY

C O R O L L A R Y.

C. When the *center*, (C) and *distance* Ci (fig. 55.) of the *picture* are given; the *vanishing line* VL with its center S and distance iS of any *original plane* whose *inclination to the picture* is known, may be found by the converse of the first part of the last problem. Thus through the *center of the picture* C, draw CS in any direction, and draw Ci perpendicular to CS, and equal to the *distance of the picture* proposed; lastly, draw iS making the angle CiS, equal the complement of the *given inclination of the picture and the original plane*, and cutting CS in S; and through S draw VL parallel to Ci, then is VL the *vanishing line* sought, S is its center, and iS is its distance, (by § 98.)

P R O B L E M XI.

CI. The *center* (C) and *distance* (B. fig. 56) of the *picture* being given; to find the *vanishing line* (MN) of all such *original planes* as are perpendicular to such *original lines* as have a given *vanishing point*, (A.)

Through the given *vanishing point* A, and the *center of the picture* C, draw an indefinite line CA; perpendicular to which draw CI, and make CI equal to the given distance B of the *picture*. 2d. Draw IA, and through I, draw IS perpendicular to IA, and cutting

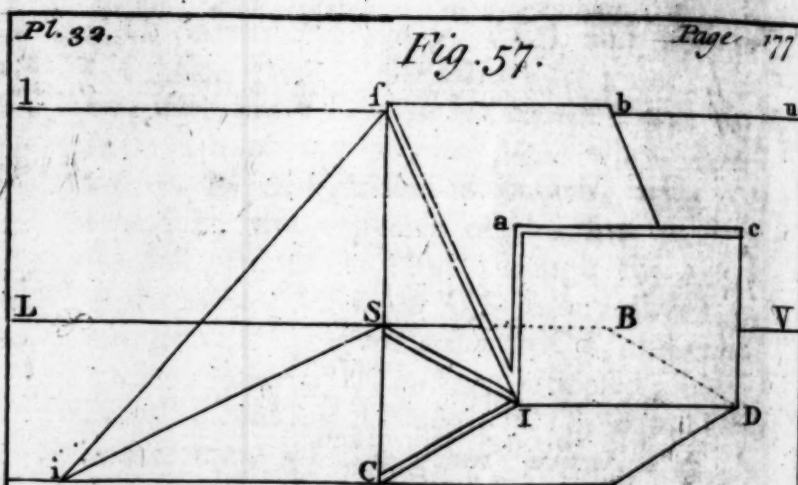


Fig. 58.

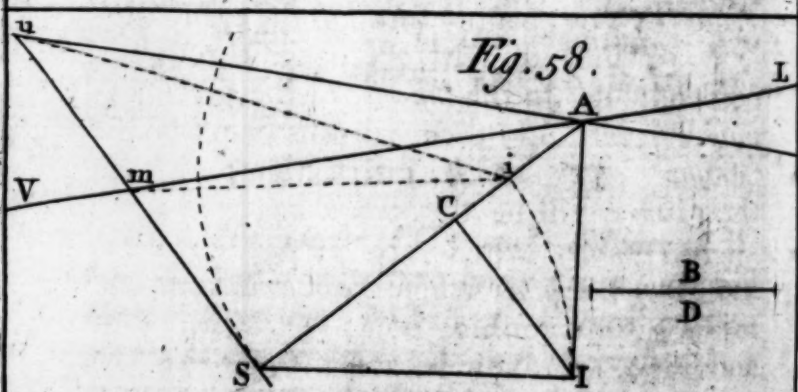


Fig. 59.



ting A C produced, in S; lastly, Draw M N through S, perpendicular to S A, then is M N the *vanishing line* sought, S its *center* and S I its distance.

DEMONSTRATION.

Let C i be a line raised perpendicular to the plane of the *picture* P (Q E N) from its *center* C, equal to its *distance*, then will i, be the *place of the eye*, or *point of sight* (by § 93). Suppose a plane S i A C to be passed by C i and through the given *vanishing point* A, then will the *distance* A I of that *vanishing point* be in that plane, which is a vertical plane (by § 95) passing through the eye i: suppose a plane i K S M, be passed through i, perpendicular to the *distance* of the given *vanishing point*, A I; then will this plane be the *vanishing plane*, and its intersection S M with the *picture* will be the *vanishing line*, of all *original planes* which are perpendicular to those *original lines* whose *representations* tend to the given *vanishing point* A, (*Vide* § 3. 4. and E. 11. 14.) Now because the plane i K S M is perpendicular to i A, its section i S with the *vertical plane* S i A C, passing by that line, will be perpendicular to A i (by E. 11. Def. 3.) Also the plane i K S M and the *picture* P Q E N are each perpendicular to the vertical plane S i A C (by E. 11. 18.) wherefore their section S M is perpendicular to the *vertical plane* S i A C, (by E. 11. 19.)

N

and

and consequently SM and AS are perpendicular to each other, and AS is the *vertical* of the vanishing line SM . (*Vide* § 96 and 97.)

So that in the solution of this Problem, we proceed as we did in § 28 and 29, by working on the *plane of the picture* instead of the *vertical plane* AS : also that S is the center and $iS = IS$, is the distance of the vanishing line SM sought, is evident from § 7.

P R O B L E M XII.

CII. *The center (C fig. 57.) and distance (CI or Ci) of the picture being given, also the vanishing line (VL) of any original plane, whose inclination to another original plane is known, and whose intersection with it is parallel to the picture: it is required to find the vanishing line, and its center and distance, of that other original plane.*

1st. Through the center C , draw CS perpendicular to the given vanishing line VL (by E. 1. 12.) and it will cut it in its center S , (*Vide* § 97.) also draw Ci parallel to VL , and equal to the given distance of the picture, and let Si be drawn.

2d. Draw if making with Si the angle Sif equal to the given inclination of the two original planes, and cutting CS , produced, in f .

Lastly. Through f , draw ul parallel to VL , which will be the *vanishing line* sought, having

having f for its center and $f i$ for its distance.

DEMONSTRATION.

Let us suppose CI (fig. 57.) to be the distance of the picture raised at right angles to it, and that $SIDB$ and $fIDb$ are the vanishing planes of the original planes in question (*Vide* § 3) intersecting the picture in the vanishing lines VD and ul , and cutting each other in the line ID which must be parallel to the picture because the intersection of their originals was supposed parallel to the picture, (*Vide* E. 11. 16.) wherefore a plane $aIDe$ may be extended by ID parallel to the picture (by § 50) then (by E. 11. 16.) VL is parallel to ID , and ul is also parallel to (ID .) consequently the vanishing lines VL and ul must be parallel to each other, and the perpendicular Cf , drawn from the center of the picture is the vertical of those vanishing lines VL , and ul , and gives the points S and f for their centers (by § 97.) wherefore the line IS and If , drawn from the point of sight I , to these centers, are the distances of those vanishing lines VL and ul , (by § 7.) Also the vertical plane fIC being perpendicular to the two vanishing planes (by § 94) must be perpendicular to their intersection ID , (by E. 11. 19.) and therefore the angle SIf is the inclination of the

vanishing planes to each other, (by E. 11. Def. 6.) which is the same as the inclination of their original planes (by E. 11. 16.) wherefore in the solution of this Problem, since Ci is drawn on the *picture* equal to CI , which is situated in the *vertical plane* CIf , and also perpendicular to the *vertical* Cf , it is evident (from the construction, and from the demonstration in § 29) that the several lines and angles drawn on the *picture* are equal to those drawn on the *vertical plane*: and must therefore cut the *vertical* in the same points S and f . Or if the *vertical plane* CfI is supposed to be turned on the line Cf till it coincides with the *plane of the picture*, the several lines drawn on it will coincide with those drawn on the *picture*.

P R O B L E M XIII.

CIII. *The center (C) and distance (B fig. 58.) of the picture being given, together with the vanishing line (VL) of any original plane, also the vanishing point (A) of its intersection with any other original plane, and the angle which measures the inclination of these two original planes to each other, being known: it is required to find the vanishing line (ul) of that original plane, and its center and distance.*

1st. By Problem 11. (§ 101.) find the vanishing line Su of those original planes which

which are perpendicular to such *original lines*, as have A, for their vanishing point, cutting the given *vanishing line* VL, in m.

2d. On the line SA (which is the *vertical* of the *vanishing line* Su) lay Si, from the center S, equal to its *distance* SI, and draw the line im.

3d. Draw iu making with im, the angle miu equal to the given inclination which the *original plane* whose vanishing line VL is given, and that whose *vanishing line* is sought, have to each other, and cutting Su in u.

Lastly, draw uA which will be the *vanishing line* sought, of which the *center* and *distance* may be found by § 100.

DEMONSTRATION.

In order to understand the reason of this proceeding, let PQEN (fig. 59) be the *plane of the picture*; C, its *center*, and CI its distance raised perpendicular to that plane; let AILD and AILD, be two *vanishing planes* intersecting each other in the line IA, and the *picture*, in the *vanishing lines* VL and ul: now since these vanishing planes are parallel to their respective *originals*, their intersection IA will be parallel to the intersections of those originals, (by E. 11. 16.) consequently A is the *vanishing point*, and IA is its *distance* of such original lines as

N 3 are

are the intersection of those *original planes*, whose *vanishing lines* are VL and ul . Again let $SIKM$ be a plane passing through the eye I , perpendicular to IA , then will each of the vanishing planes $AILD$ and $Aild$ be perpendicular to this plane $SIKM$ (by E. 11. 18.) and their intersections Im and Iu with it, will be perpendicular to IA , (by E. 11. Def. 3) and the angle mIu will therefore measure the inclination of the vanishing planes $AILD$ and $Aild$ to each other, (by E. 11. Def. 6.) but the lines Im and Iu being situated in the plane $SIKM$, will cut the *picture* somewhere (as at m and u) in the *vanishing line* SM of that plane; also because the said lines Im and Iu are in the *vanishing planes* $AILD$ and $Aild$, they must cut the *picture* somewhere in the *vanishing lines* VL and ul , consequently the lines Im and Iu pass thro' the points where SM cuts the *vanishing lines* VL and ul .

Let IS be drawn along the plane $SIKM$ perpendicular to the vanishing line SM then will S be its center and SI its distance (by § 7.) and we may now observe that if on any plane that passes along the vanishing line SM (among which the plane of the picture must be reckoned) a perpendicular Si (fig. 58. and 59.) be drawn through S , equal to SI the *distance* of the said *vanishing line*, and if from this point i , lines are drawn to the
points

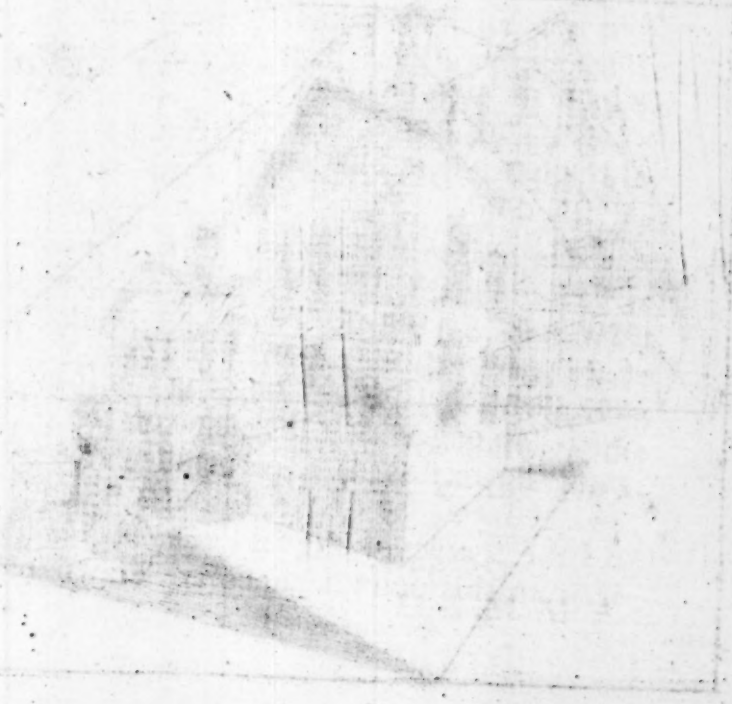
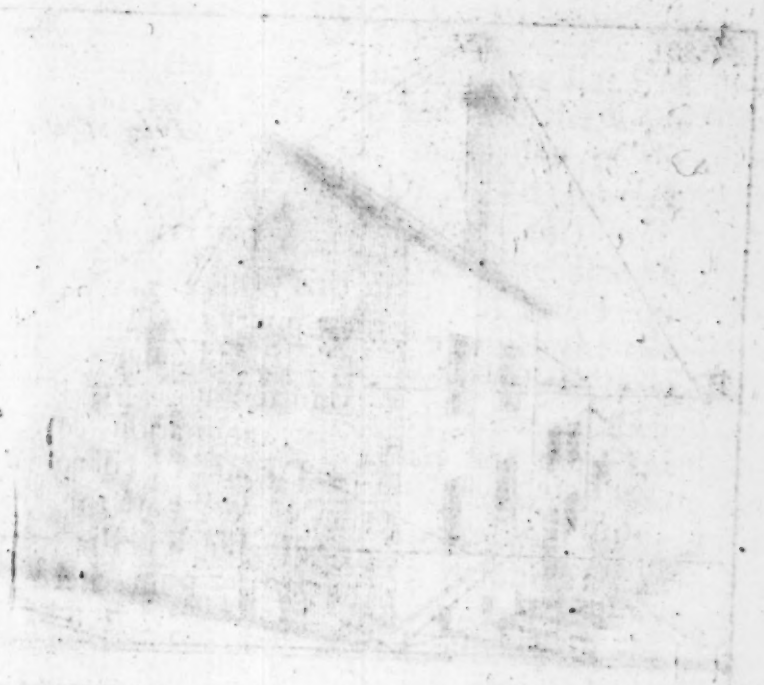


Fig. 60.

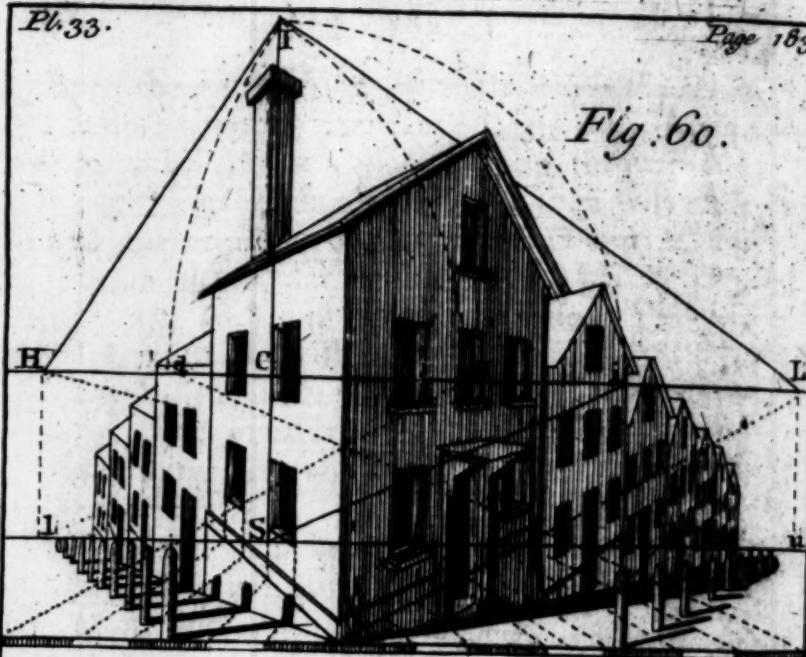
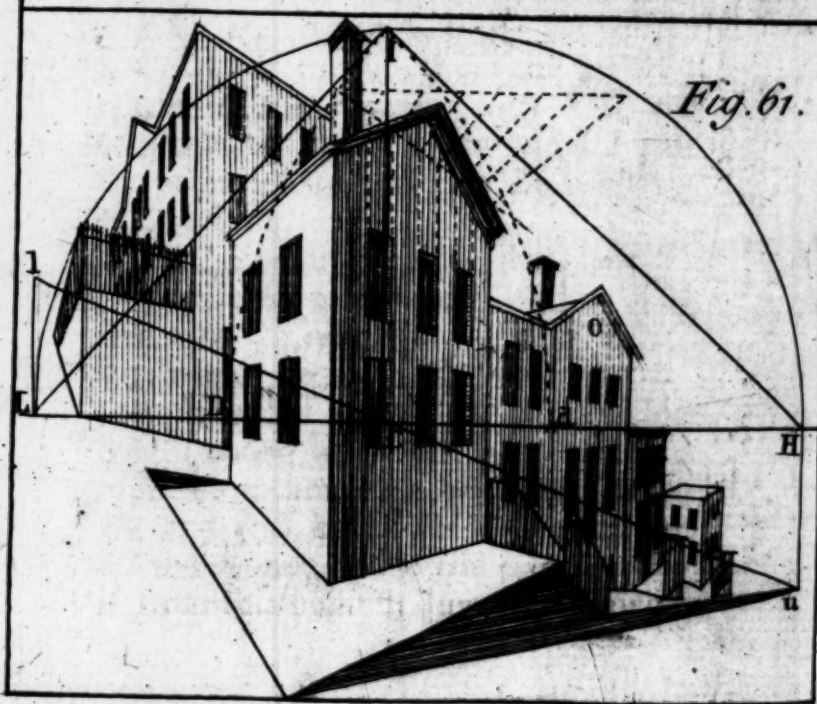


Fig. 61.



points m and u where the vanishing line SM cuts the two others VL and ul , the angle $m i u$ will measure the inclination of the vanishing planes $AILD$, $Aild$ to each other, (*Vide the demonstration of § 29.*)

If then we have one vanishing line as VL (fig. 58. and 59.) given, and know the inclination $m i u = m I u$ which the two *vanishing planes* have to each other, (that being the same as the inclination of their respective originals) we may find the point u , through which the other *vanishing line* must pass as the problem directs; and as the point A in which the two vanishing lines cut each other is given, a line ul passing through those points A and u must be the vanishing line sought.

In the most extensive practice and most intricate subjects that can be proposed to exercise the art of perspective, I believe there are none that will embarrass my reader if he has made himself master of these 13 problems; which indeed comprize the whole art, with the most necessary aids and abbreviations that are to be found in any of the voluminous books on this subject: and it is on account of their great utility that I have been so prolix in their proof and explication. I shall add a few examples to illustrate the 4 latter problems contained in this chapter.

L E M M A.

CIV. *In an upright picture the vanishing lines of all planes perpendicular to the horizon, will be perpendicular to the horizontal line.*

D E M O N S T R A T I O N.

For the *vanishing plane* of such *original planes* as are perpendicular to the horizon will be perpendicular to the *vanishing plane of the horizon*; also the picture when upright is perpendicular to the *vanishing plane* of the horizon, wherefore the lemma is manifest by (E. 11. 19.)

E X A M P L E XIV.

Of a Down-hill View.

Figure 60 is the *representation* of two rows of houses standing on a hill which *descends* from the picture. In representations of this kind we must suppose the plane of the hill to be cut by a plane parallel to the horizon, and that the situation of this section with respect to the picture, is given or assumed. If this section is supposed parallel to the picture (as in the present case) and if we know the angle C i S of the hills declivity, its vanishing line u l, which will be parallel to the horizontal line H L may be found by problem 12, as I shall shew.

The

The picture is supposed perpendicular to the horizon, wherefore its center C and distance C I is also the center and distance of the *horizontal line*, H L. We must next remember that although the bottoms of the houses cannot be parallel to the horizon, yet the tops of the windows, and of the houses, if we suppose them flat roofed, will be parallel to the horizon, and therefore will tend to some point in the *horizontal line*, H L. Let L be assumed for the *vanishing point* of all lines which are parallel to the horizon and situated in the right-hand front: then will H be the *vanishing point* of all lines that are parallel to the horizon and situated in the left-hand front (*Vide* page 104.) Lay the *distance of the picture*, C I, on the *horizontal line* from C to i; draw C S perpendicular to the *horizontal line*, and S i making with it the angle C i S equal to the declivity of the hill, and cutting C S in S. Through S draw u l parallel to H L, which shall be the *vanishing line* of the hill. (*Vide* prob. 12, page 178.) Through H and L draw L u and H l perpendicular to H L, and cutting u l in the points u and l, which are the *vanishing points* of all lines situated in the fronts of the buildings and parallel to the ground; for the *vanishing lines* of the fronts must pass through H and L; wherefore by § 104, Lu and Hl are these *vanishing lines*. But the *vanishing points* of all lines situated in these fronts must
be

be in these *vanishing lines*, (by § 13.) and if they are parallel to the ground they must also tend to some point in its *vanishing line*, $u l$; wherefore u and l are the *vanishing points* of all lines that are situated in the fronts and parallel to the ground. All the other operations will appear from inspecting the scheme and reflecting on what has been already taught. But it may be proper to observe that D is the distance of the *vanishing point H ; and d that of L laid on the *horizontal line*.*

EXAMPLE XV.

Of a Down-hill and an Up-hill View.

In fig. 50. $H L$ is the *horizontal line*, C its center, and $C I$ its distance, which are also the center and distance of the picture. The section of the plane of the hill by a plane parallel to the horizon is supposed perpendicular to the picture, and therefore C , (the center of the picture) is its *vanishing point*. According to this supposition the *vanishing plane of the hill* and the *vanishing plane of the horizon* being each perpendicular to the picture, the intersection of the picture with these planes will measure their inclination to each other (by E. 11. Def. 6.) that is the angle $H C u$ formed by the *vanishing line* of the hill, $u l$, and the *horizontal line*, $H L$, is equal to the declivity of the ground. Wherefore through C , draw $u l$, making, with the *horizontal line* $H L$, the angle $H C u$, equal to the proposed steepness of the hill. The point



Fig. 62.

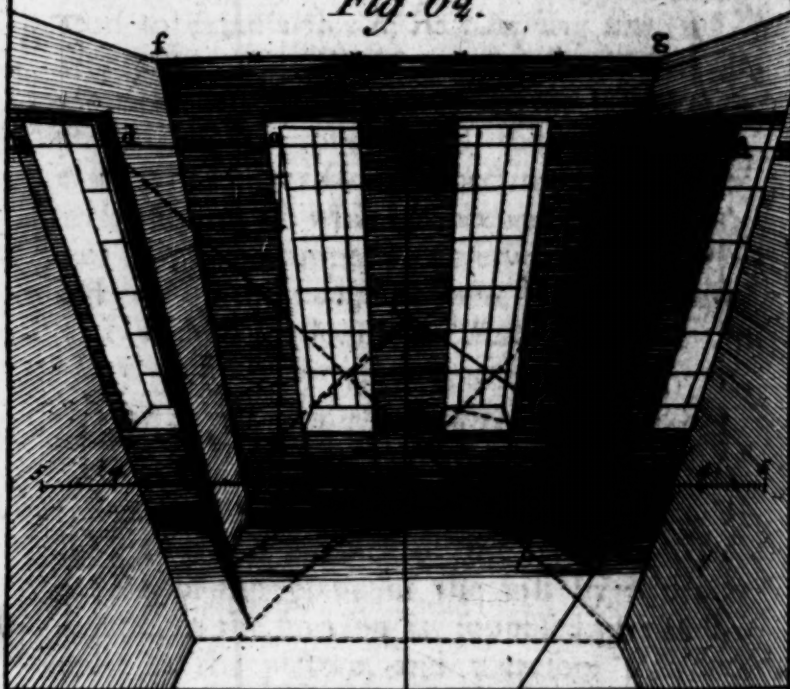


Fig. 64.

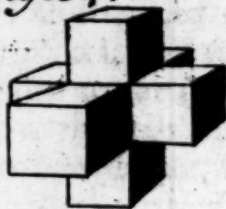
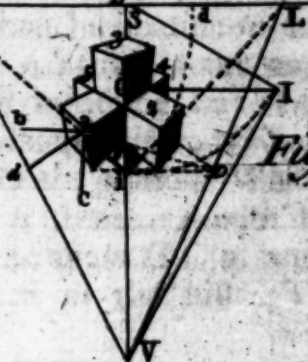


Fig. 63.



point L is assumed for the vanishing point of such lines as are parallel to the horizon and situated in, or parallel to, the right-hand fronts of the buildings. By this the vanishing point, H, of lines parallel to the horizon and situated in, or parallel to, the left hand fronts are found. Also the points u and l are found by Lemma, § 104, whose uses are the same as in the last example. Also d is the distance of the vanishing point L, and D is that of H, laid on the horizontal line. The supposed unevenness of the ground make the several horizontal platforms, steps, &c. which are represented in the scheme, necessary; whose construction, it is presumed, will appear by inspection.

N. B. If the section of the hill, by an horizontal plane had not been supposed parallel to the picture, as in the last example, nor perpendicular to it, as in this, we must have had recourse to problem 13. § 103, in order to find u l the vanishing line of the hill.

EXAMPLE XVI.

Of the inclined picture.

In fig. 62. we have the representation of the inside of a room. The picture is supposed to be inclined to the ground, for which reason the horizontal line will not pass through its center, but will be at a greater or less distance from it, according as the picture is more or less inclined to the ground, and it will pass above or below the said center,
according

according as the picture inclines from, or towards the eye, as will be evident from the least reflection. To draw the representations of objects upon such a plain we proceed as follows. The point C is assumed for the *center of the picture* in any convenient part of it, Ci is made equal to the intended distance of it from the eye, (*Vide* § 93.) or for want of room Ci may be made equal to $\frac{1}{2}$ &c. of that distance. Next (by § 100.) we find the *vanishing line* H L, and its center and distance of a plane inclined to the picture so much as we have determined the picture shall be inclined to the ground, (*Vide* § 92.) then will this line H L be the *horizontal line*. The vanishing point of such original lines as are perpendicular to the horizontal plane must next be found (by § 99.) and then we are sufficiently prepared to draw the perspective representation of any building, &c. on our inclined picture, the appearance of which, though uncouth and distorted if the picture is viewed in an erect position, will be just and natural when inclined to the horizon so much as was intended, and the eye of the spectator is placed exactly in the position and at the distance pitched upon when the representation was drawn. But to be more particular, C is the center of the picture, and Ci is half its distance, because we have not room on the plate to lay down the whole distance; the angle Cfi is made equal to the
the

the inclination which the plane of the picture is supposed to have to the ground; (*Vide* § 92.) wherefore *f* would be the center of the horizontal line, if *Ci* was the whole distance of the picture, but because it is only half the distance, *CS* must be made equal to twice *Cf*, and *S* will be the center of the horizontal line. (*Vide* E. 6. 4.) In like manner if *Ci* were the whole distance of the picture, *V* would be the *vanishing point* of all lines which are perpendicular to the ground; (by § 99.) but the true *vanishing point* of such perpendicular lines, will be twice the distance of the point *V* from *C*, which would exceed the bounds of our plate; wherefore we proceed as taught in § 82, dividing *VB* each way from *V* into equal parts two of which are equal to one of the parts which *Ci*, produced both ways, is divided into.

On the *horizontal line*, I lay from its center (*S*) *Sd*, equal to *fi* which is half its distance, also we make *Vm* equal to *Vi*, and *CD* equal to twice *Cm*; then is *D* the true dividing center of all lines perpendicular to the horizon, (*Vide* page 119.) One side *fg* of the ceiling is in this example supposed parallel to the picture, on this side we fix the situation of the windows and draw their perpendicular sides, also the up and down sides of the frames by means of the divisions on the lines *Ci* and *VA*. (*Vide* § 82) the horizontal
frames

frames are formed by means of the line on , and the dividing center D ; on , being made of any length in proportion to the width of the window which was before fixed. Those sides of the cieling and floor which are *not* parallel to the picture are perpendicular to those which are, and therefore must tend to the center of the horizontal line. The doors are stationed and made of any determinate width in proportion to the width of the windows, by means of the half distance point d (*Vide* prob. 5. page 110.) The doors being open are supposed to have L , or any point at pleasure for their vanishing point, the distance $L\Delta$ of which is found by means of the distance SI of the horizontal line, SI being made equal to twice fi , and $L\Delta$ is made equal to LI ; from whence we obtain the representation of the doors: a view of the figure, and the skill which we hope the reader has by this time obtained, render it needless to expatiate farther.

E X A M P L E XVII.

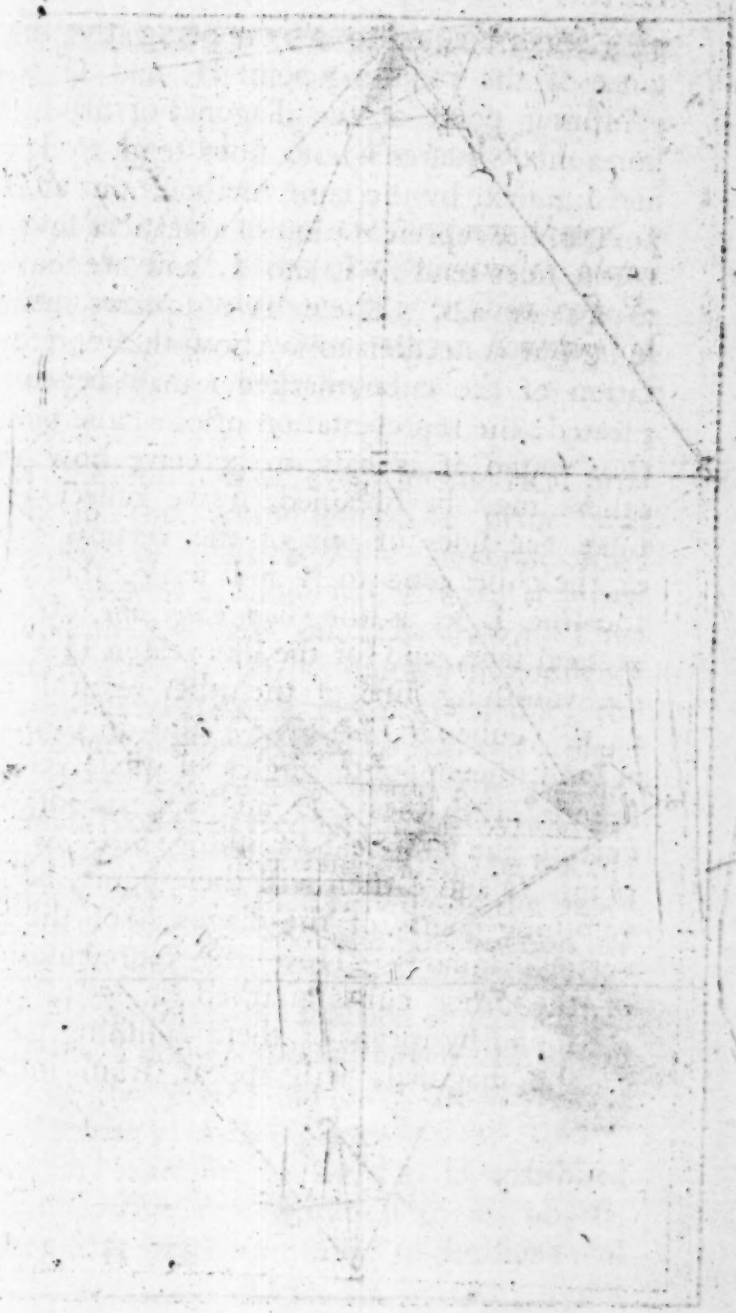
On an inclined Picture.

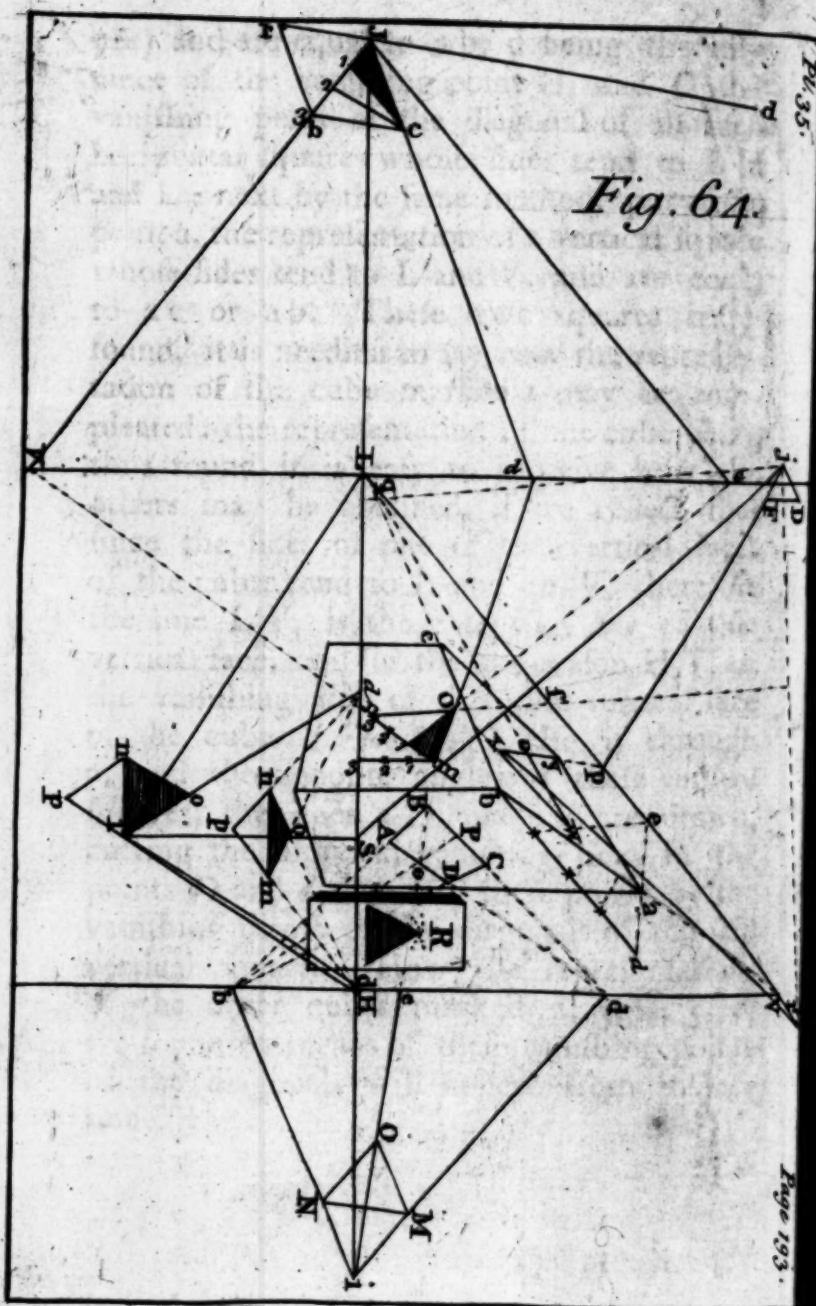
Fig. 62 is the representation of a *double cross* on a picture inclined to the horizon: the double cross is a figure composed of 6 equal cubes placed upon each of the faces of a 7th cube, which may therefore be accounted as the center of the whole figure. From this account of the double cross,

(of

of which to render it more intelligible we have given a view in fig. 63. as it would appear on an upright picture) it is plain that the center cube will always be entirely hid because surrounded with all the others; supposing that the reader understands the nature of the object to be represented, let us return to fig. 62, C is the center of the picture, and CI its distance: The horizontal line HL is drawn by § 100. of which S is the center. SI its distance, and SV its vertical; then by § 99. find the vanishing point V, of such lines as are perpendicular to the horizon. Lay the distance SI of the horizontal line, on its vertical from S to i, and assume any point H in the horizontal line for the vanishing point of two sides of the horizontal faces of the cubes which compose the double cross intended to be represented; then will iL drawn perpendicular to Hi give on the horizontal line the point L, for the vanishing point of the other two sides of the horizontal faces of the said cubes. Let us now assume any point a, for the nearest upper corner of the nearest cube, from this point draw a b parallel to the horizontal line HL and a c, parallel to its vertical VS, and let each of them be assumed, or made (by § 65.) equal to the intended length of a side of the cube, Next at the point a, put the representation of an horizontal square, whose sides tend to H and L (*Vide* Examp. 2. page 98)

98.) and are equal to $a b$, d being the distance of the vanishing point H , and D the vanishing point of the diagonal of all such horizontal squares whose sides tend to $L H$ and L ; next by the same method, put at the point a , the representation of a vertical square whose sides tend to L and V , and are equal to $a c$ or $a b$. These two squares being found, it is needless to say how the representation of the cube marked 1 may be completed: the representation of one cube being thus found it is easy to perceive how the others may be obtained, if we reflect that since the sides of one of the vertical faces of the cubes tend to L and to V , therefore the line $L V$, is the *vanishing line* of that vertical face, and for the like reason $H V$, is the vanishing line of the other vertical face of the cube: (*Vide* § 13) also if through a , and the opposite angles of these vertical squares, the lines $a D$ and $a d$ are drawn, cutting the aforesaid vanishing lines in the points D and d then will these points be the vanishing points of the diagonals of the said vertical squares. How the representations of the other cubes marked 2, 3, 4, 5, 6, are found by means of these vanishing points of the diagonals will appear from inspection.





C H A P. IX.

A farther application of the foregoing theory, to the solution of geometrical problems on planes situated in every variety of position respecting the picture: containing a summary of the practice of perspective.

IN the solution of every geometrical problem in the introduction, a plane was supposed on which the several lines given, as well as those which were required to solve the problem, might be drawn. Thus when in E. r. i. It is proposed to construct an equilateral triangle on a given line, the practitioner is supposed to be allowed a plane surface as a slate, paper, board, the ground, &c. on which the given line is drawn; and the other two sides of the triangle are to be traced; and such a plane, which answers to the paper on which the schemes of the introduction are drawn, may be termed the *working-plane*.

But in the following solution of geometrical problems, I do not suppose any *working plane* really to be given, but only the *perspective representation* of such a plane.

As for example, in fig. 64, is exhibited the perspective representation of an object, whose visible parts are similar to the sides and

roof of a house, as also the plane of the ground it stands on, which are all supposed oblique to the picture. There is also exhibited the *representation* (R) of a plane standing upright on the ground and parallel to the picture.

Now any of these planes may be considered as a *working plane*, on which it may be required to perform any geometrical problem contained in the introduction: but on all this variety of planes there are only two different ways of working; for every plane whose *representation* is given, must be either parallel to the picture, as the plane R; or it must be oblique to it, as are all the others.

When the *working plane*, whose *representation* is given, is parallel to the picture, every problem is solved exactly in the same manner as taught in the introduction; because the *representations* of all figures on such planes are similar to their originals. (by § 70) But if the *working plane* is oblique to the picture, its *vanishing line*, and its *center* and *distance*, must be given, or found by § 100 from the known *center* and *distance* of the picture, and the *inclination* of the said *working plane* thereto.

In the present fig. 64. H L. is the *horizontal line*; whose *center* is S, and *distance* SI; which, because the picture is supposed erect, is also the *center* and *distance* of the picture. L is the *vanishing point* of such lines as are parallel to the left hand front of the object and to the ground:

ground: wherefore the line $V L$ drawn through L perpendicular to $H L$, is the *vanishing line* of the said front by § 164; also $S L$ is its *vertical* (by § 97) L its *center* and $L I$ its *distance* (by § 98). Let this *distance* $I L$ be laid from L to J on the *vertical* $H L$, continued; then may J be esteemed as the *place of the eye* (*Vide* page 24) for finding all representations that are to be exhibited on the plane of the said left hand front. In like manner $V H$ is the *vanishing line* of the right-hand front, $H S$ is its *vertical*, H its *center*, and $H I$ its *distance*; which *distance* being laid from H on the said *vertical* produced, gives the point i for the *place of the eye* to be used in finding *perspective representations* on the said right-hand front. Let us now search for the *vanishing line* of the roof. We have given the *representations* $a b$ and $b c$ of the edges of the roof, but these lines $a b$ and $b c$ are also situated in the *fronts* of the object as well as in the roof: wherefore the *vanishing points* of these lines must be somewhere in the *vanishing lines* $V H$ and $V L$ of the said fronts, (by § 87) now $a b$ being produced cuts $V H$ in V ; and $b c$ being produced cuts $V L$ in L ; wherefore V and L are the *vanishing points* of $a b$ and $b c$, consequently since we have got the *vanishing points* of two lines situated in the roof; a line $V L$ drawn through these points, will be the *vanishing line* of the roof, (*Vide* § 89) a perpendicular to which drawn

from the center of the picture (S) cuts it in its center f, and gives Sf for its vertical, on which its distance fj (being to be found by (§ 100) may be laid from f to j, then will j be the place of the eye for finding representations on the plane of the roof. (Vide page 24.)

We are now furnished with all the requisites for performing any geometrical problem on any of these planes; neither of which is more difficult than another, and all are subject to the same rules: so that any problem which I perform on the plane of the ground, may with the same facility be performed on either of the fronts, or on the roof, and although (because it might be difficult for the learner to divest himself of all local ideas at once) I have supposed the picture erect; whereby the representations of the corners of the object are perpendicular to the horizontal line, which makes the situation of its parts more intelligible at first sight: yet this is only to be considered as accidental; for in the next plate, by making the picture inclined to the horizon, every thing will be in the most oblique state possible; but the trouble of solving any problem will not be at all enhanced thereby; for every thing will be performed with the same ease as we find the representation of a line on the ground in an erect picture.

P R O B L E M I.

The representation of a line being given to divide it in parts whose originals are equal, or bear any assigned proportion, to each other, (Vide E. 6. 9 and 10.)

Let us consider the roof of the object, fig. 64. as the working plane, whereof VL is the vanishing line, whose center is f , and distance fj , and let its edge ab be the given line, which we will suppose requires to be divided into 4 parts equal in representation. Through one end b of the given line ab , draw bd , parallel to the vanishing line VL , and on it, lay 4 equal divisions as expressed by the dots, lay a ruler by d and a , cutting the vanishing line VL in e , then the ruler laid by e and the divisions on bd will divide ab into 4 parts equal in representation.

Vide this problem with its demonstration page 123 § 84.

N. B. If the given line had been parallel to the vanishing line, the problem would have been solved exactly the same as E. 6. 9. (Vide § 70.)

P R O B L E M VII.

The representation of a right line being given, on it to construct the representation of an equilateral triangle. (Vide E. 1. 1.)

When the working plane is parallel to the picture, as R, fig. 64.

The problem is then solved in all respects like E. 1. 1. (which sees) for the representation of an equilateral triangle, is an equilateral triangle (by § 70)

And because when the working plane is parallel to the picture, the solution of any problem is the same as taught in the elements, we shall for the future omit this case.

C A S E II.

When the given line is parallel to the picture, and the working plane is oblique.

Let the working plane on which the problem is to be solved be the ground (fig. 64.) (whereof the vanishing line is H L, whose center is S and distance S I) on which let m n be the given line parallel to the picture, and to the horizontal line. (Vide § 44.) Through I, the place of the eye, draw a line parallel to H L, and on it erect an equilateral triangle, (by E. 1. 1.) parallel to whose sides draw through the point I, the distance lines (Vide § 9.) I d and I d, cutting the horizontal line in d and d, then through the extremities m and n of the given line draw lines

lines to the *vanishing points* d and d , cutting each other in o , then is the figure mno the *representation* of an equilateral triangle as required.

The reason of this and most of the proceedings in this chapter may be gathered from § 34. page 25. where it is shewn that when any *original figure*, and the *place of the eye*, are laid down on the plane of the picture as taught in page 23. the *vanishing points* of the sides and diagonals of the said *original figure*, are found by drawing parallels to those sides and diagonals through the *place of the eye*. If then instead of having the original figure $ABCD$ (fig. 8. page 25.) drawn in its proper place, below the *vanishing line*, (*Vide* page 23.) we draw a figure $Ibcd$ at I , similar to it, and also situated so that its sides may be parallel to the sides of the *original figure* $ABCD$, whose representation is wanted. It is evident that the same *vanishing points* will be obtained by producing the sides of this figure $Ibcd$, or drawing parallels to them through the place of the eye (I) as if we made use of the *original figure* $ABCD$, and that if we have the representation ab , of one side of the *original figure* given or assumed, the rest of the *representation* may be determined without having any recourse to the said original figure $ABCD$.

OBSERVATION

It will be proper to remember that when the place of the eye (I, fig. 8.) as taught in page 23 § 33. and when one angle of the similar figure I b c d is placed at the eye (I) and the representation a b c d is determined thereby. The right and left hand parts of it, will correspond with the right and left hand parts of the representation. Also those parts of it which are nearest the vanishing line V L, will correspond with those parts of the representation which are nearest the vanishing line. Thus b, (fig. 8.) which is the right hand angle of the similar figure I b c d corresponds with b, which is the right hand corner of the representation a b c d; also c, which is that corner of the similar figure I b c d, nearest to the vanishing line, (V L) corresponds with c, which is that corner of the representation nearest to the vanishing line.

C A S E III.

When the given line and the working plane are both oblique to the picture.

Let the given line m n (fig. 64.) be situated in the left front of the building, whose vanishing line is V L, having L for its center, and L J for its distance, already determined.

Produce

Produce m till it cuts the vanishing line $V L$ in V , and draw $V J$ which is the distance of the given line $m n$. (*Vide* § 33.) On this distance line $V J$ construct an equilateral triangle $J b c$ (by E. 1. 1.) Through J draw lines $d J$ and $d' J$ parallel to the sides $b c$ and $J c$ of this equilateral triangle, and where they cut the vanishing line $V L$ of the working plane in d and d' , will be the vanishing points of the two sides $m o$ and $n o$ whose representation is sought, which may be drawn as in the last case.

N. B. When the given side $m n$ of any triangle is oblique to the picture, it will generally happen that one of the sides whose representation is sought will be so nearly parallel to the picture that its vanishing point will not fall within the compass of the paper, as in the present case the vanishing point of $n o$, whose distance is $J a$, falls much beyond the bounds of the plate. To remedy inconveniences of this kind, divide the given representation $m n$ into two or three equal parts (by prob. 1. of this chap.) also divide that side $J b$ of the triangle $J b c$, which is the distance of the said given line $m n$ into the same number of equal parts (by E. 6. 9.) and let these divisions on $m n$ and on $J b$ be numbered the same way, (in general it will be sufficient to divide them each into two equal parts, as in this example, and then there is no occasion for numbering.) Draw a
line

line $c i$ through the first division on $j b$ and the opposite angle of the triangle $I c b$, and through J draw a line parallel to it, and cutting the vanishing line $V L$ in e , then through the first division on $m n$, draw the *visual* $i e$, cutting the visual $m d$ in o . Lastly, draw $n o$, which compleats the *representation* of the triangle sought, and if produced would cut the vanishing line $V L$ in the same point as $J d$ would cut it if produced.

The truth of this practice will be evident from the observation page 200. from whence it appears that the figure $m o i$ and $n o i$ represent triangles similar and correspondent to $J c i$ and $b c i$, in all their parts.

OBSERVATION.

When the representation of any line is given as a base for drawing the representation of any proposed figure, it ought to be mentioned or determined whether the given line is to be esteemed as the nearest or most remote boundary thereof. Thus if the line $m n$ situated on the ground plane (fig. 64.) and parallel to the vanishing line $H L$ is to be considered as the nearest side of the equilateral triangle which is to be represented, the other sides $m o$ and $n o$ will fall nearer to the vanishing line, wherefore the similar triangle $I n o$ at the place of the eye must be formed with the corresponding sides $I o$ and $n o$ towards

wards the vanishing line: but if the said given line $m n$ is to be considered as the most remote side of the equilateral triangle which is to be represented, it is evident that the other sides $m p$ and $n p$ must not be drawn towards the vanishing line; and the similar equilateral triangle $I p n$ must be formed with the sides $I p$ and $p n$, which correspond to those sides whose representation is sought, receding from the vanishing line $H L$. (*Vide* page 200.)

And although the same vanishing points d and d are found by either of the equilateral triangles $I n o$ or $I n p$, yet in drawing of more complex figures, much perplexity will ensue if this consideration is neglected: for it is the triangle $I n o$ which corresponds with the representation $m n o$; and though $I n p$ will give the same vanishing points, it does not correspond with the triangle represented by $m n o$, to which it is in a situation entirely reversed.

In the same manner although the equilateral triangle $J b r$ will give the vanishing points for determining the representation $m n o$ on the left hand front of the object (fig. 64.) yet it is not that triangle, but $J b c$ whose situation agrees with the equilateral triangle represented by the said $m n o$ as found by case 3. page 200.

PROBLEM

P R O B L E M III.

At any point in the representation of a plane, to put the representation of a line equal to one whose representation is given, (Vide E. 1. 2.)

Let the right hand front of the object (fig. 64.) be the working plane, whose vanishing line is VH , whereof the center is H , and the distance is $H i$ (Vide page 195.) also let A , be a point, at which it is required to put the representation of a line equal to that represented by BC .

Produce BC , till it cuts the vanishing line VH in d , also draw Ad . Through one extremity (B) of the given line (BC) and the given point A , draw the visual BA cutting the vanishing line in b , lastly draw Cb cutting Ad in D . Then is AD the representation of a line equal to that represented by BC .

For BC and AD represent parallel lines because they tend to the same vanishing point d , (Vide § 17.) as do the lines AB and CD , wherefore the figure $ABCD$ is the representation of parallelogram, and AD and BC must represent lines equal to each other, by E. 1. 34.

P R O B L E M IV.

The representations of two lines being given: to cut off from the greater a part equal to the less, (Vide E. 1. 3.)

C A S E

C A S E I.
When one of the given representations is parallel to the vanishing line of the working plane, and the other is not.

Let the working plane be the roof of the object, in fig. 64. whose vanishing line is $V L$, whereof f is the center and $f j$ is its distance; $w x$ is the representation of the shorter line which is parallel to the vanishing line $V L$, and $x y$ is the representation of the longest line, which if produced cuts $V L$ in p , (which is its vanishing point by § 88.) Draw $p j$, also draw $D j$ parallel to the vanishing line $V L$, and on these lines take $D j$ and $F j$ equal to each other; and draw $D F$; also through the eye j draw $j q$ parallel to $D F$. Lastly, Join w and q , cutting $x y$ in o , then does the part $x o$, represent a line equal to that represented by $w x$. For a little reflection on the observation, page 200. will shew, that the triangle represented by $w x o$ is similar and correspondent to the triangle $D j F$, which is an *isocelas triangle*, by construction.

C A S E II.

When neither of the given lines, are parallel to the vanishing line of the working plan.

This case is performed on the right-hand front in (fig. 64.) of which $V H$ is its vanishing line, H its center, and $i H$ its distance

tance. (*Vide* page 195.) The given lines are BC and AB , whereof AB is the *representation* of the least line, and it is required to cut off from BC , a part that shall represent a line equal to that represented by AB .

Produce the given lines to their vanishing points b and d , and draw their distance lines ib and id , on these lines take iM and iN equal to each other, and draw MN , which because it is almost parallel to the vanishing line VH , its *vanishing point* will fall too remote for our purpose; wherefore I compleat the rhombus $iMON$, (*Vide* E 1. def. 32) by drawing MO and NO parallel to the distance lines id and ib , and I draw the diagonal io , which I produce till it cuts the vanishing line VH , in e . The problem is now easily solved, for I draw a line AD , through A to the vanishing point d , and cut it in o by a line drawn through B and e ; lastly a line drawn through b and o , cuts BC in P , then does BP represent a line equal to that represented by AB . $Q.E.D.$

For its evident that the figure $ABPo$ is the *representation* of a rhombus, similar and corresponding to $iMON$, whereof all the sides are equal.

P R O B L E M V.

The *representation* of a right-lined angle being given, to draw the *representation* of a line,

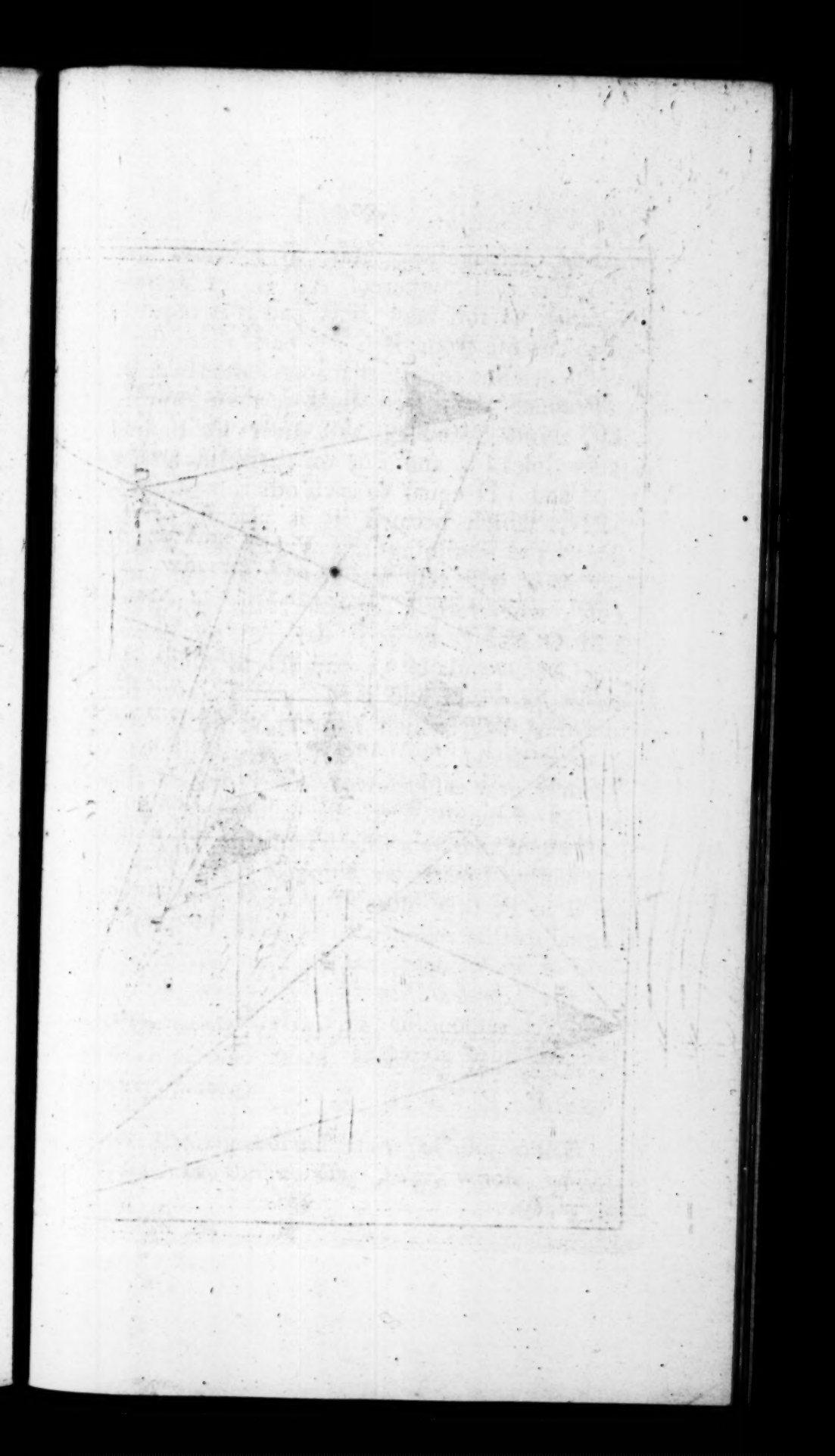
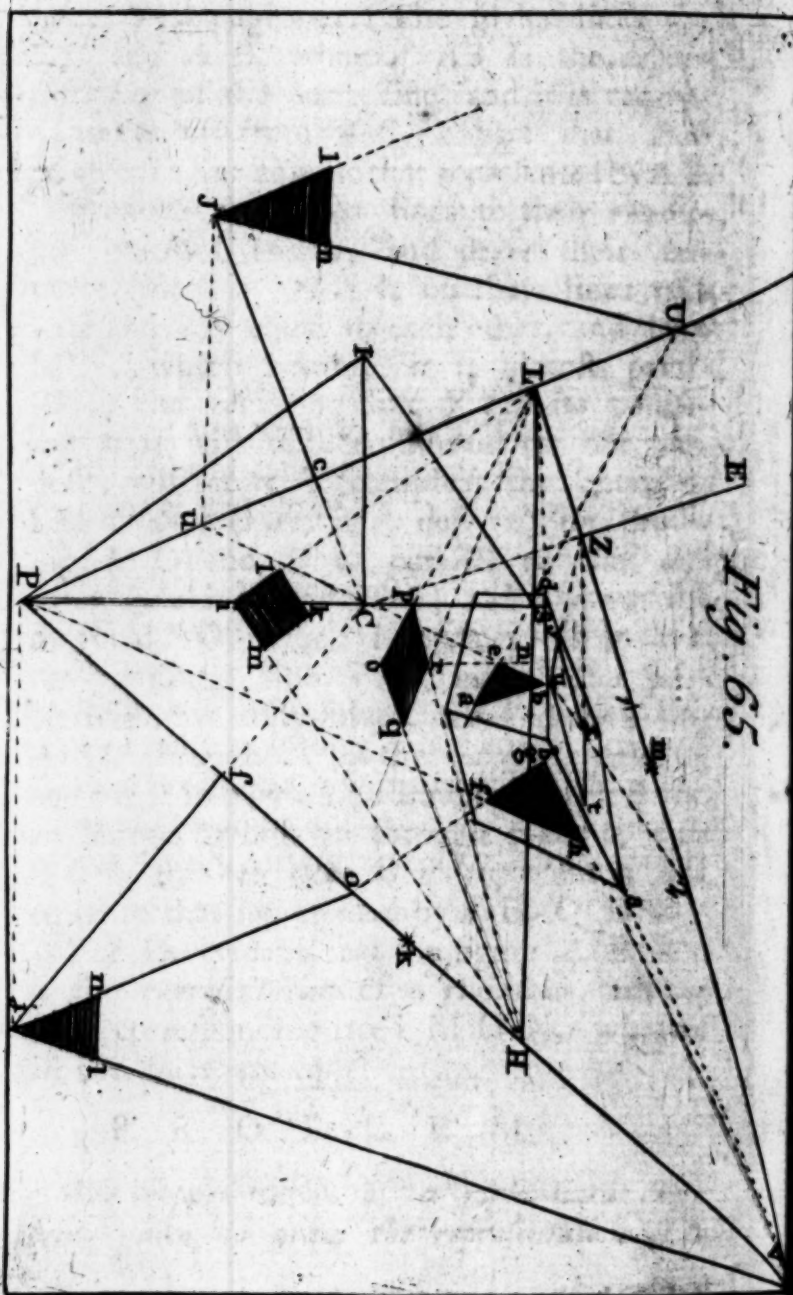


Fig. 65.



line, dividing it into two equal parts.
(Vide E 1. 9.)

C A S E I.

When one of the sides forming the given angle, is parallel to the vanishing line of the working plane.

Let the roof of the object (fig. 64) be the working plane, whose vanishing line is VL , whereof f is the center, and fj is its distance, and let the given lines forming the angle be wx and xy , whereof wx is parallel to the vanishing line VL .

Produce xy to its vanishing point p , and draw the distance line jp . Also through j draw jD parallel to the vanishing line VL . Lastly, draw jr (by E 1. 9.) bisecting the angle Djp , and cutting the vanishing line in r . Then a line from x to r , will be the representation of a line bisecting the angle whose representation (wxy) was given. Vide § 19. 28. 32. and 33.

C A S E II.

When neither of the sides forming the given angle is parallel to the vanishing line of the working plane.

Let the right-hand front of the object (fig. 64.) be the working plane, whose vanishing

ishing line is VH ; H its center; and iH its distance (*Vide* page 195.) Also let AB and BC be the representations of sides of the given angle.

Produce AB and BC to their vanishing points, b and d ; also draw the distance lines, ib and id ; draw (by E. 1. 9.) the line ie bisecting the angle $b id$, and cutting the vanishing line in e ; lastly draw the line Be , which will be representation of the line sought. (*Vide* § 19. 28. 32. and 33.)

Preparation of the working planes exhibited in Fig. 66.

Seeing fig. 64. is become full of lines, in fig. 65. there is the same kind of object as in 54, but the picture which was before supposed perpendicular to the ground, is now inclined to it; whereby every part of the said object is oblique to the picture.

C is assumed for the center of the picture, and CS is drawn at pleasure for the vertical of the horizontal line: (*Vide* § 100.) perpendicular to this, CI is drawn and made equal to the proposed distance of the picture. Then IS is drawn, making with CI the angle CIS , equal to the complement of the angle which the picture is to make with the ground
the

the point (S) where this line cuts C S, is the center of the horizontal line H L, which must be drawn through S, and parallel to C I, or perpendicular to its vertical S C. On S C produced, lay S i, equal to S I, the distance of the horizontal line H L; then will i, be the *place of the eye* for solving all problems, when the working plane is supposed parallel to the ground (*Vide* § 28. Next (by prob. 10. page 173) find the *vanishing point* P, of such lines as are perpendicular to the plane of the ground, (whereof the vanishing line is H L,) let L (or any point in the *horizontal line*,) be assumed for the *vanishing point* of such lines as are parallel to the ground and to the left-hand front of the object; (whose faces are to be the working planes in the ensuing problems,) then H i drawn perpendicular to L i gives H for the vanishing point of the lines that are parallel to the ground, and to the right hand front. Also those boundaries or corners of the said figure that are perpendicular to the ground must tend to P. Since then all lines drawn in the left-hand front and parallel to the ground, have L for their vanishing point, and since all lines drawn on the said front perpendicular to the ground have P for their vanishing point; it follows (by § 89, page 169) that P L is the *vanishing line* of the said left-hand front. In like manner it is evident H P

P is

is the *vanishing line* of the right-hand front. Having thus found the vanishing lines of the fronts, their centers and distances are easily determined by § 100. page 176. Thus C f, drawn through C the center of the picture, perpendicular to H P, gives f for its center. Also from f lay S k on H P equal to C I the distance of the picture: then is C k the distance of the vanishing line H P, which being laid from f to j, gives the place of the eye for solving problems when the right-hand front of the object (or any other plane parallel to it) is considered as a *working plane* (Vide § 28.) In like manner c is the center of L P and j is the place of the eye for solving problems when the left-hand front or any of its parallels are considered as a *working plane*. The right-hand boundary a b of the roof of the object, being also situated in the right-hand front, its *vanishing point* must be somewhere in the vanishing line H P. (by § 87, page 168) wherefore producing a b, it cuts H P in V, which is its vanishing point: seeing then that V is the vanishing point of a b, and that L is the vanishing point of b d (which is the left hand boundary of the roof, and is parallel to the ground) it follows by (§ 89 page 169) that V L is the *vanishing line* of the said roof. A perpendicular (C Z) to which, drawn through C, (the center of the picture) gives Z for the center of V L. Also if on V L, I lay from Z, Z m equal to C I, the *distance of the picture*; m C would be the distance

distance of $V L$; which if laid from Z on the vertical $C Z$ continued, would give the place of the eye for solving problems on the plane of the roof or any of its parallels. (Vide § 28.) But because I have not room on the plate to do this, I lay on $C Z$ continued, $Z E$, equal to half $m C$, the true distance of $V L$. Then I may esteem E as the place of the eye, if I remember that every vanishing point I obtained by so doing, will be but half as far from the center Z as it ought to be. Also every dividing center I lay on the said vanishing line must be made twice as far from Z , as it falls when I consider E as the true place of the eye (Vide E. 6, 4.)

P R O B L E M VI.

The representation of a line being given, through any point to draw the representation of a line perpendicular to it. (Vide E 1. 11. and 12.)

Let the working plane be the ground (fig. 65.) whereof $H L$ is the vanishing line and i , the place of the eye (vide § 28.) also let $o p$ be the given line and q the given point.

Produce $o p$ to its vanishing point L ; draw the distance line $i L$, and draw $i H$ perpendicular to it: (by E. 1. 11) then through H , and the given point (q) draw $q H$. which will be the representation of a line perpendicular to that represented by $o p$.

It were needless to speak of the reason of this operation which has so often been explained before, or to observe that if the given line $o p$ had been parallel to the vanishing line $H L$, the representation of a line perpendicular to it would have had S , the center of the vanishing line, for its vanishing point. (Vide § 21, page 12.)

P R O B L E M VII.

Through any given point to draw the representation of a line that is inclined to a line whose representation is given, in a given angle. (Vide E 1. 23.)

The solution of this problem differs nothing from that of the last, for if the line $o q$, was not to represent a line perpendicular to that represented by $o p$, instead of drawing $i H$ perpendicular to $i L$, we must have drawn it (by E 1. 23.) so that the angle $H i L$ might have been equal to the angle which the line whose representation is given, makes with that whose representation is sought. Vide § 19 and 30.

P R O B L E M VIII.

The representation of a line being given, through any point to draw the representation of a line parallel to it.

Let the plane of the ground fig. 65. be the working plane, and through the point q , let it

it be proposed to draw the representation of a line parallel to that represented by o p.

Produce o p, to its vanishing point L, then through the given point q draw q L which will represent a line parallel to that represented by o p. (Vide § 17.)

If o p had been parallel to the vanishing line, q r must have been drawn through q parallel to o p (Vide § 54 and § 6.)

P R O B L E M IX.

The representation of a line being given on it, to draw the representation of a triangle, similar to a given triangle.

C A S E I.

When the given line is parallel to the vanishing line of the working plane.

Let the left hand front of the figure, fig. 65 be the working plane, whereof L P is the vanishing line, c its center, and J the place of the eye, Vide § 28: also let a b (in the left hand front) be the given line. Through J draw J l, at pleasure, parallel to the vanishing line L P, and on it (by E 6 18.) construct a triangle J l n similar to that required to be represented, taking care to place its angles (by the observation page 206) in the same situation respecting J l as you would have them in respect of the given line a b. Thus, because I would have the largest angle at b.

(which is the right-hand end of a b) I make the largest angle at l: (which is the right-hand end of J l) also because I would have the vertical angle (e) fall towards the *vanishing line* L P, I make the vertical angle n towards the *vanishing line* L P.

Produce J n till it cuts the *vanishing line* in U: also through J, draw J u parallel to n l, and cutting the *vanishing line* L P in u. Lastly, through the extremities of the given line a b draw *visuals* tending to u and U, and crossing each other in e. Then is a b e the representation of a triangle similar to the triangle J l n. Vide § 28, 32 and 34.

C A S E II.

When the given line is not parallel to the vanishing line of the working-plane.

Let the right-hand front of the object, fig. 65 be the *working plane*; whereof the *vanishing line* is V P, whose center is f, and the place of the eye is j (Vide § 28) and let the given line be f g; produce f g to its vanishing point o, draw its *distance* j o, and on it construct a triangle j n l similar to that whose representation is required (by E 6. 18.) produce j l to its vanishing point V, also through j draw j P parallel to l n. and cutting the *vanishing line* V P in P. Lastly through f and g draw the *visuals* g V and f P cutting each other in h. then is f g h the representation of a triangle similar to j l n, whereof j corresponds

[illegible]

The first of these is the fact that the
 government has been unable to raise
 the necessary funds to meet its
 obligations. This is due to a
 combination of factors, including
 the high cost of borrowing and
 the low level of tax revenue.
 The second factor is the
 government's failure to implement
 effective fiscal policies. This has
 led to a large and growing
 budget deficit, which has in turn
 contributed to the country's
 economic problems.

PL. 37.

Page 115.

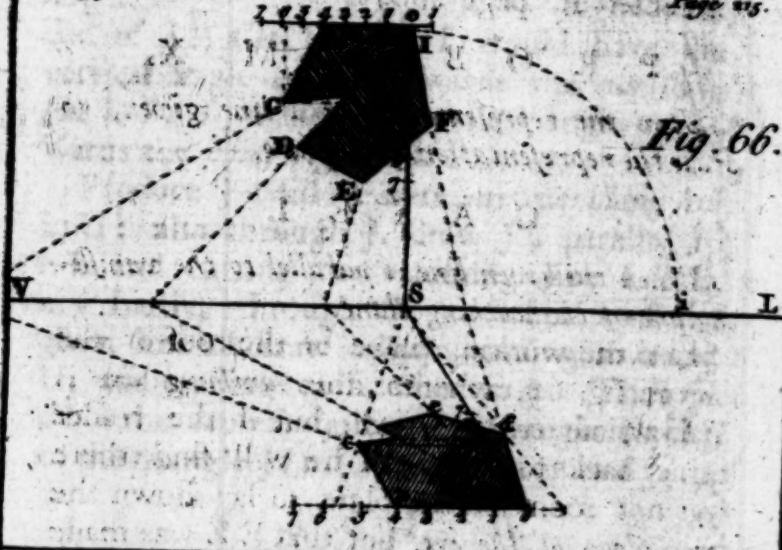


Fig. 66.

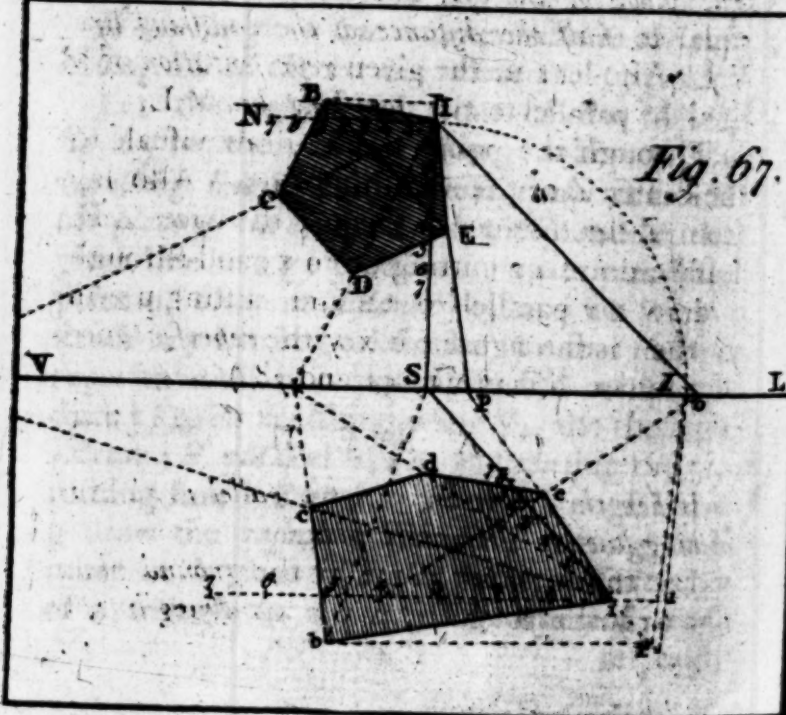


Fig. 67.

responds with g, n with f, and l with h.
Vide observat. page 200)

P R O B L E M X.

Upon the representation of a line given, to draw the representation of a square.

C A S E I.

When the given line is parallel to the vanishing line of the working plane.

Let the working plane be the roof of the object fig. 65, whereof the *vanishing line* is V L whose center is Z; but if the reader turns back to page 211 he will find there was not room on the plate to lay down the *true place of the eye*, but that E Z was made equal to *half the distance of the vanishing line* V L. Also let t u, the given representation of a line, be parallel to the *vanishing line* V L.

Through the points t and v draw visuals to the center Z of the vanishing line. Also lay from Z the distance Z 2 equal to twice Z E. Lastly, draw u 2 cutting t z in x; and through x draw x y parallel to t u and cutting u z in y, then is the figure t u x y the representation of a square (*Vide § 21, 54 and 78.*)

C A S E II.

When the given line is not parallel to the vanishing line of the working plane.

Let the working plane be the ground-plane fig. 65. whereof H L, is the *vanishing line*, S

its center; and *i*, the place of the eye, Vide § 28. And let *o p* be the representation of a line, upon which it is required to draw the representation of a square.

Produce *o p* to its vanishing point *L*, and draw its distance line *i L*: on this line at *i*, make a small square (by E. 1. 46.) observing the observation, page 200. produce its side *i m*, to *H*, and draw the diagonal till it cuts the vanishing line *H L* in *n*. Then through the extremities *o* and *p* of the given line, draw lines *o q* and *p r* tending to *H*, also draw *o n*, cutting *p r* in *r*. Lastly draw *r L*, cutting *o q*, in *q*: then is the figure *o p q r* the representation of a square. For it is evident the two triangles *o q r* and *o p r* are the representations of triangles similar to the triangles *i m k* and *i l k*. Vide § 34 and page 200.

P R O B L E M XI.

The representation of a line being given, on it to construct the representation of a right-lined figure, similar to one given.

C A S E I.

When the representation of the line given is parallel to the vanishing line.

Let *V L* fig. 66, be the vanishing line of the working plane (which may either be parallel or ever so oblique to the ground or horizontal

horizontal plane) and let S be its *center* and SI its *distance*. Also let $a b$ be the representation of a line parallel to the vanishing line VL , on which it is required to draw the representation of a figure similar to $IB C D E F$.

Through I let IB be drawn parallel to the vanishing line, on which by E 6. 18. let the figure $IB C D E F$ be drawn similar and in a like position with regard to IB , as the figure required to be represented is in regard to the line represented by $a b$. Resolve this figure into triangles and find their representation by problem 9 of this chapter.

Another Method.

When the figure to be represented is complex and consists of many sides, the last method will be very puzzling and troublesome. But the following, which is a little improvement of De Sargue's method of reticulation, will render the most irregular multangular figures almost as easy to be represented as the more simple and regular.

The similar figure $IB C D E F$ is constructed as before. I divide the side IB which corresponds with the given line $a b$ into any number of equal parts, and continue the divisions and number them each way from I as far as necessary. I also divide the given line $a b$ into the same number of equal parts and continue the divisions and
number

number them in like manner. Then through the point *a*, I draw a line (*Sa*) to the center of the vanishing line.—Next I lay on the perpendicular *IS*, the same divisions as are on *IB*, and continue them till they exceed the greatest length of the similar figure *IB C D E F*, and number them as the figure shews. Also laying the distance *SI* on the *vanishing line* from *S* to *i*, I lay a ruler by *i* and the several divisions on *a b* and by this means obtain the divisions on *Sa* which are to be numbered so as to correspond with the divisions on *IS* (*Vide* page 200.) we are now furnished with all the necessaries to determine the representation required.

Thus if I want to find the representation of the point which corresponds to *C* in the similar figure *IB C D E F*, through *C*, I draw a line parallel to *IS*, and I observe it cuts *IB* at about $5\frac{1}{4}$, wherefore through $5\frac{1}{4}$ on the line *a b*, I draw a line tending to *S*. Again through *C*, draw a line parallel to *IB*, and I observe it cuts *IS* at about $3\frac{1}{2}$ wherefore through $3\frac{1}{2}$ on the line *Sa*, I draw a line parallel to *a b* and it cuts the line before drawn, in *c*, which therefore is the representation of that point which answers to *C* in the similar figure *IB C D E F*. In the same manner may the representation of any other point be found, and the lines joining such points, will be the representations of the corresponding sides of the said similar figure.

The

The reason of this proceeding is too obvious to need pointing out, and the learner will find by practice, that sometimes the representation of a side will be best determined by the former, and sometimes by the latter method, so that by blending them properly together, he will shorten his labour and prevent embarrassment.

C A S E II.

When the given line is not parallel to the vanishing line.

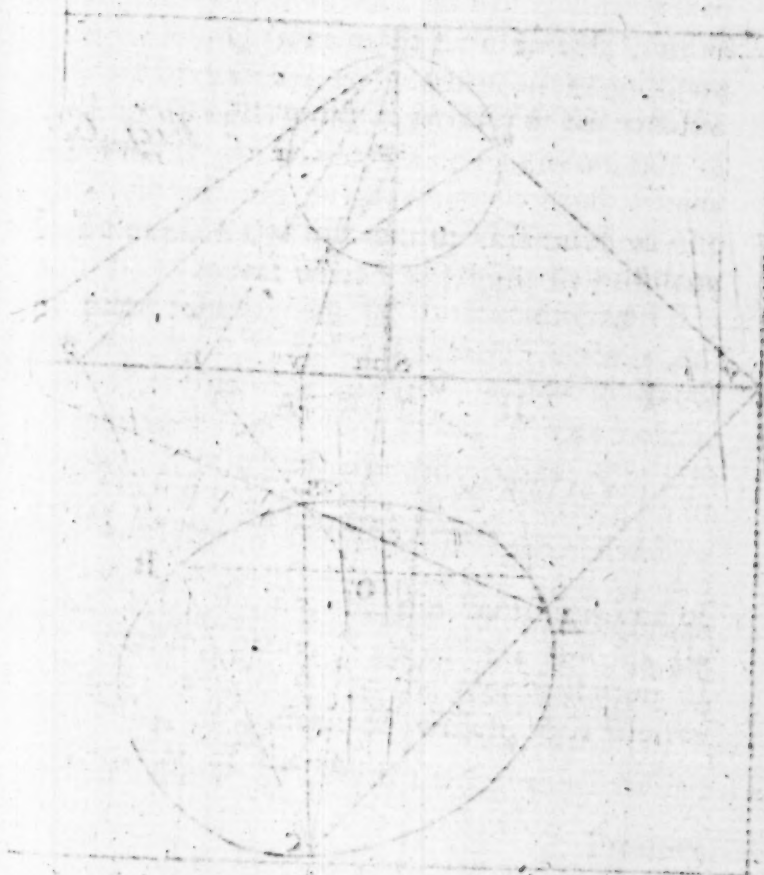
Let VL (fig. 67.) be the vanishing line of any working plane of which S is the center and SI the distance; also let ie be the representation of a line, on which it is required to draw the representation of a regular pentagon (or any other figure.) I produce ie to its vanishing point p . and draw Ip . on which line I construct a small regular pentagon $IBCDE$ (having regard to the observation page 200.) This being done I resolve it into triangles whose representations may be found by problem 9 of this chapter.

Another Method.

2. The regular pentagon $IBCDE$, being constructed on the distance line Ip , as before, through I , I draw a line IN , parallel to the vanishing line VL , and on it, and also on IS (which is perpendicular to it) I lay any number of small equal divisions, and number them

them each way, from I, as the figure shews: through E (which corresponds to the other extremity [a] of the given line) and any division on the line I N (as the fourth) I draw E 4, and through I, draw I o parallel to E 4, and cutting the vanishing line in o, then draw n i through i, parallel to V L, and cut it in the point n, by a line passing through e and o; so shall i n e be the representation of a triangle similar to I 4 E; and because I 4 consists of 4 parts, I divide n i into 4 equal parts, and continue the divisions each way from i, and number them in a manner corresponding to the numbers on I N: next I lay the distance I S on the vanishing line from S to I, and drawing i S, a ruler laid by I, and the several divisions on n i give me the divisions on i S, which I number so as to correspond with those on S I.

The representation of any angular point of the similar figure I B C D E, may now be found by the same means as in the last case, (page 218) Thus to find the representation of the point B; through the point B, I draw a line parallel to I S, and it cuts I N at about $4\frac{2}{3}$, wherefore, through $4\frac{2}{3}$ on the line n i, I draw b S; also through B, a line drawn parallel to I N, cuts I S at about $\frac{1}{4}$ of a division on I S, above the point I, wherefore through the point r, which is about $\frac{4}{5}$ of a division, on i S, below the point i (Vide page 200) I draw r b, parallel to V L cutting



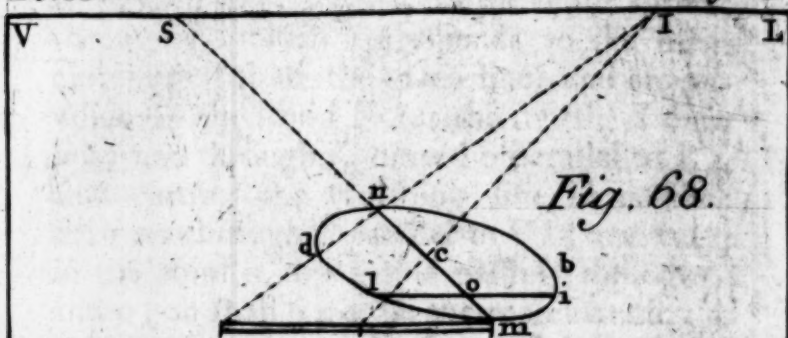


Fig. 68.

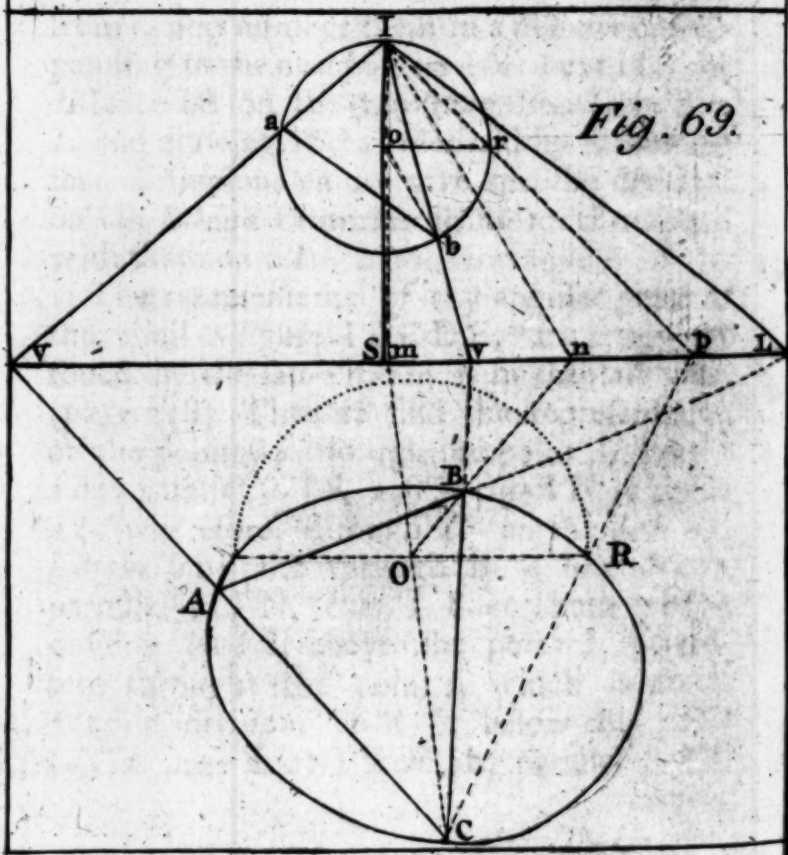


Fig. 69.

cutting bS in b , which is the representation of the point which corresponds to B , in the similar figure $I B C D E$. In like manner may the representations of the other angular points be found.

P R O B L E M XII.

The representation of a line being given as a radius, to draw the representation of a circle.

C A S E I.

When the given line is parallel to the vanishing line.

To explain the solution of this case, would only be to repeat what was taught in problem 9, page 127.

C A S E II.

When the given line is not parallel to the vanishing line of the working plane.

Let VL , fig. 68, be the vanishing line of any working plane, S its center, and SI its distance; also let ci be the representation of the radius, of a circle of which, c is the representation of the center.

Produce

Produce ci to its *vanishing point* V , and draw its distance IV : on any point of which (as C) as a center with the radius, CI describe a circle (N. B. *Had the nearest end* (i) *of the given line* ic , *been considered as the center, the point* I *must have been made the center of this circle.* Vide page 200.) through C , draw the diameter DB parallel to the *vanishing line* VL ; and also through c draw db parallel to VL . Draw IB , and produce it till it cuts VL in e , and draw ie cutting db in b . Lastly, at c with the radius cb , describe a semi-circle, and considering cb as a radius parallel to the *vanishing line*, find the representation of the circle by problem 9 of chapter 7, page 127.

For it is evident, that cbi is the representation of a triangle, similar and corresponding to CBI (Vide page 200) wherefore cb is the representation of a line equal to the given radius ci .

It is easy to perceive, that if in the circle IBD any right line figure is described, and if, by the last problem the representations of the angles of that figure are found, they will form so many points through which the *ellipsis*, representing the proposed circle, must pass.

P R O-

P R O B L E M XIII.

The representation of a circle being given to find the representation of its center.

Let $n d m b$ (fig. 68) be the representation of a circle, situated in a plane, whose vanishing line is $V L$, whereof S is the center, and $I S$ the distance. Through any point i , in the given curve, draw a line parallel to the vanishing line, and cutting it again in the point l . Bisect this line il in the point o (by E 1. 10.) and through o draw $o S$ cutting the given curve in the points m and n . Lastly, by problem 1. of this chapter, divide $m n$ into two parts $n c$ and $m c$ equal in representation, and the point c is the representation of the center of the circle represented by the ellipse $n d m b$.

For il represents a chord parallel to the vanishing line by § 54. also $m n$ is a perpendicular passing through the middle (c) of it (by § 21.) and therefore is a diameter by E 3. 3. the middle of which (c) must be the center.

P R O B L E M XIV.

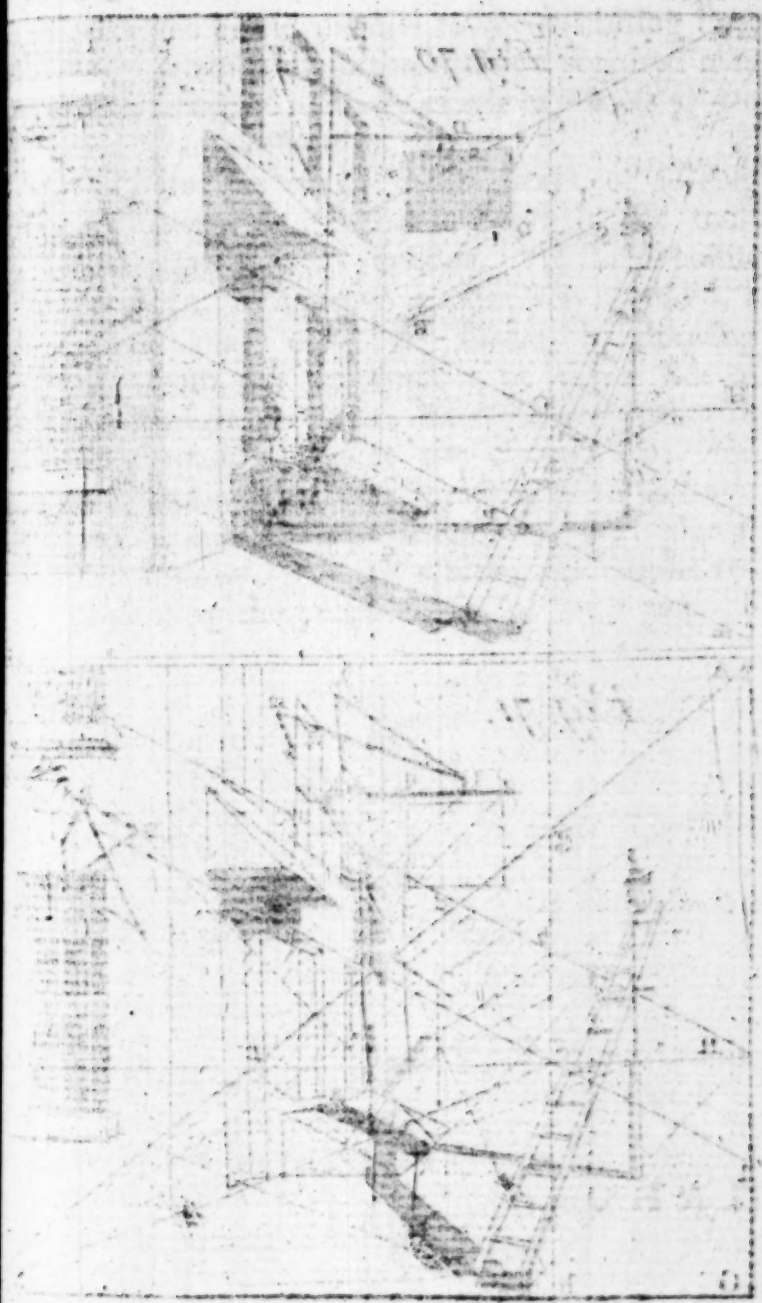
Through any three points (not situated in a right-line) in any working plane, to draw the representation of a circle.

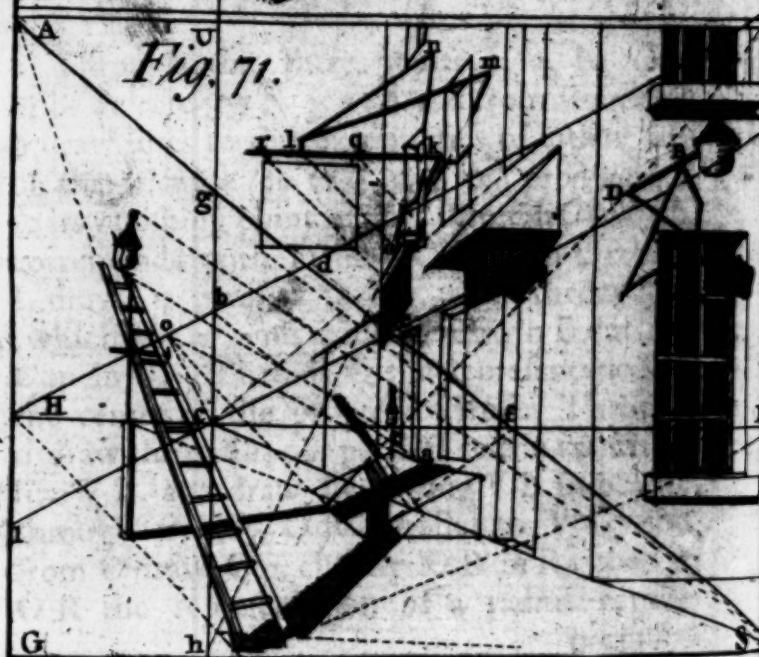
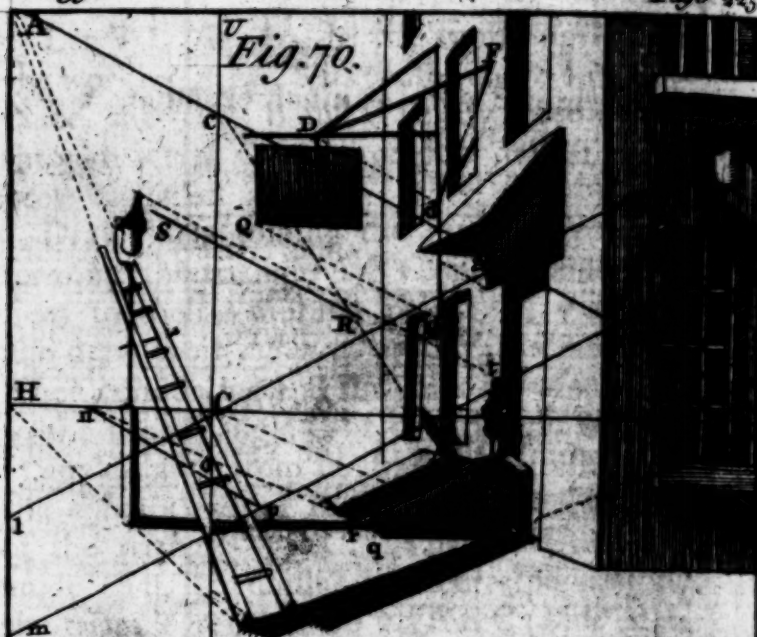
Let

Let $V L$ (fig. 69) be the vanishing line of the working plane, S its center and $S I$ its distance, and let A, B and C be any three points through which it is required to draw the representation of a circle.

Draw lines joining the given points and forming the triangle $A B C$ and produce the sides to their vanishing points V, v and L , also draw their *distance lines* $I V, I v$ and $I L$. Then considering I as the point corresponding to the nearest angle C of the triangle $A B C$; at any distance from it, draw $a b$ parallel $I L$, the *distance line* of the most remote side $A B$ of the triangle $A B C$. Then will the triangle $I a b$, be similar and corresponding to the triangle represented by $A B C$ (*Vide* page 200.)

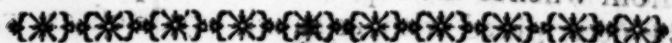
Through the three points I, a, b , describe a circle (by E. 3. 25.) and from its center o draw lines to the extremities of either side of the triangle (as to I and b .) produce $I o$ to its vanishing point m , and through C (which corresponds with I) draw $C m$, also through I draw $I n$ parallel to $b o$, and through B (which corresponds with b) draw $n B$ cutting $C m$ in O . Then is O the representation of the center of the proposed circle. Through o draw the radius o, r parallel to the vanishing line $V L$, and draw $I r$ cutting $V L$ in p . also through O draw $O R$ parallel to $V L$, and from C draw $C p$, cutting $O R$ in R : then is $O R$ the representation of a radius of the proposed





proposed circle parallel to the vanishing line, from whence the representation required may be determined with as much exactness as we please, by problem 9. page 127.

There are many other ways of solving this problem, but the above is the most exact and not very tedious. As three points are given, if no great exactness is required, a fourth may easily be found, by drawing lines from the extremities of either side of the triangle I a b, to intersect each other in the circumference of the circle I r b; and then finding the representations of those lines, which, intersecting each other, will give a fourth point, through which, the ellipsis representing the proposed circle, must pass.



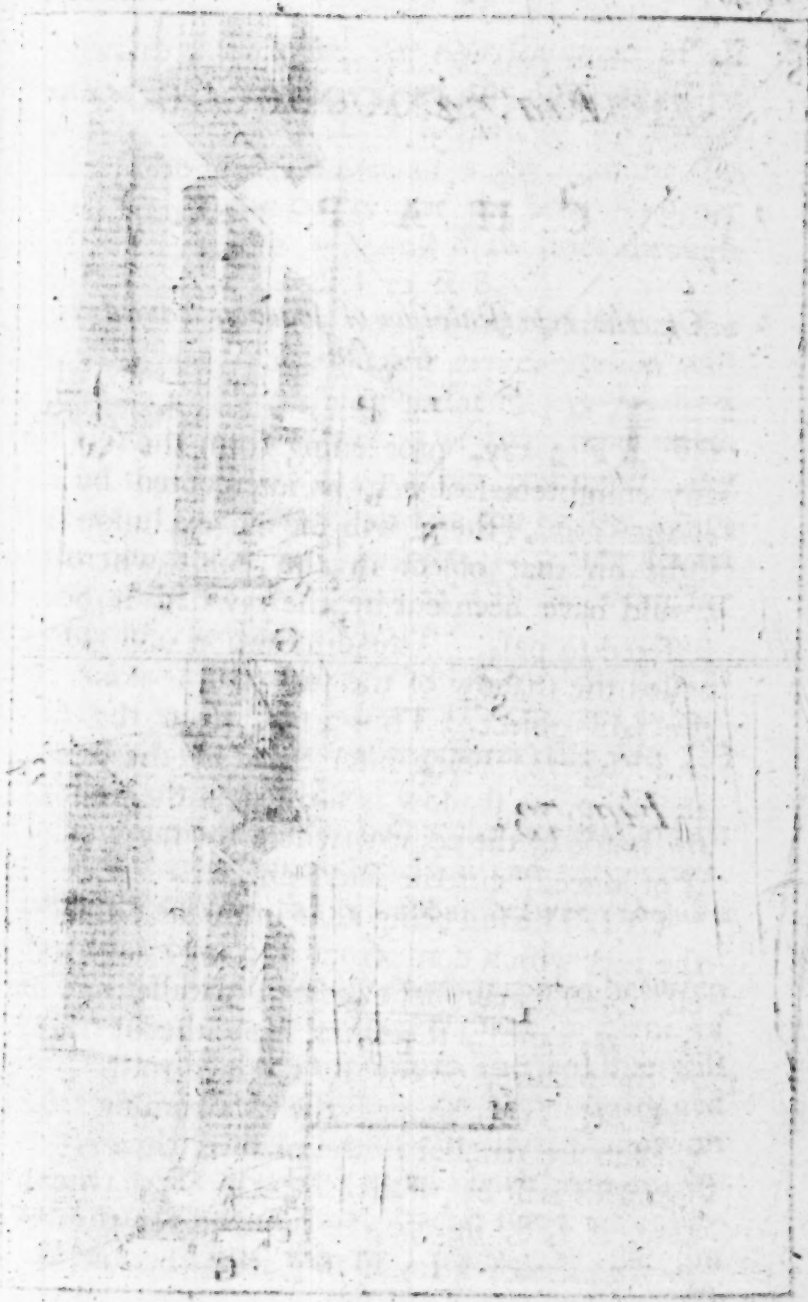
C H A P. X.

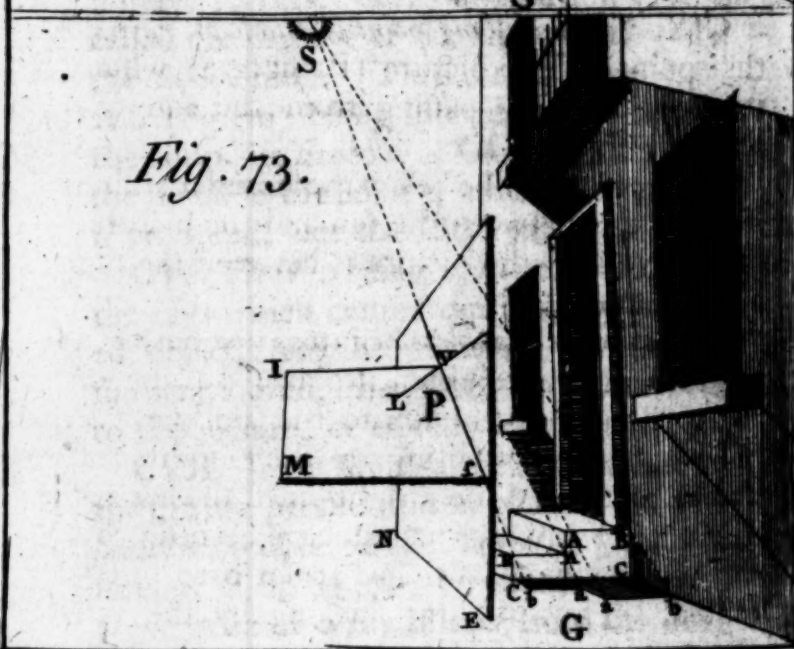
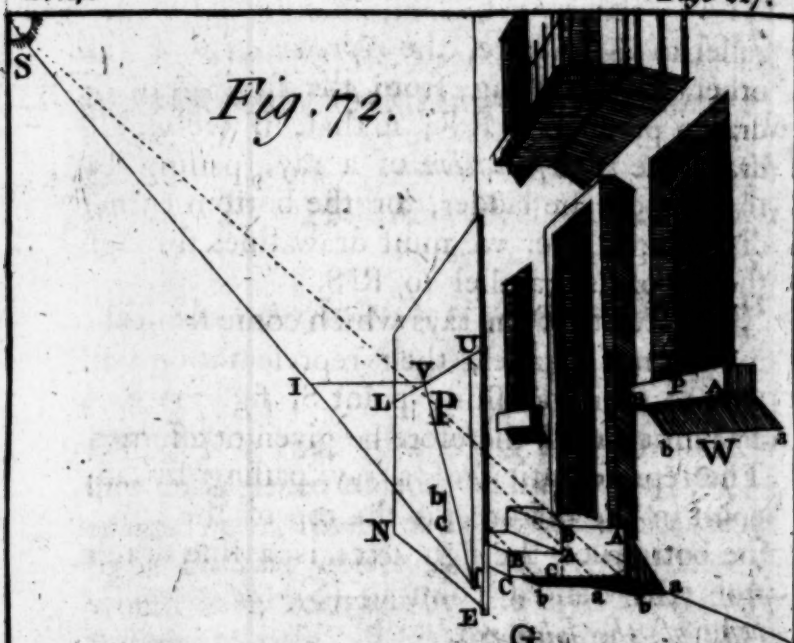
Of the representation of shadows caused by the sun.

CV. IF a ray, proceeding from the sun, to any enlightened object, be intercepted by an *opaque point*, there will be an exclusion of light on that object in the point where it would have been cut by the ray had it been suffered to pass. This diminution of light is called the shadow of the said opaque point on the said object. Thus a ray, from the sun at S (fig. 70) being intercepted by the top of the lamp, its shadow is formed on the front of the house in the point t, where the ray would, if produced, cut the said front.

CVI. The sun being at an immense distance, the rays which come from it to any enlightened object, may be esteemed parallel. The sun's rays must, therefore, be either parallel to the picture, or equally inclined to it.

CVII. If we suppose the rays coming from the sun are parallel to the picture, their *representation* will be parallel lines (by § 53) wherefore, if R S, fig. 70, be assumed for the *representation* of a ray issuing from the sun, parallel





rallel to the picture, the *representations* of all other rays coming from the sun, must be drawn parallel to R S; so that, if we would draw the *representation* of a ray, passing by the top of the ladder, or the bottom corner of the sign, &c. we must draw lines through those points parallel to R S.

CVIII. But if the rays which come from the sun are not parallel, their representation will tend to their vanishing point S, fig. 71. by § 17. which must therefore be given or assumed. The representation of a ray passing by any point in the picture, as the top of the lamp, the bottom of the sign, &c. is a line drawn from the desired point to (S) *the vanishing point of the solar rays.*

CIX. *The vanishing point of the solar rays* is that point on the picture (P, fig. 72) where it is cut by a ray passing from the sun (S) through the eye (I.)

This point will be below the horizontal line if the eye is between the sun and the picture, but above it if the picture is between the sun and the eye.

CX. If any *opaque line* be interposed between the sun and an enlightened plane, a plane of rays passing from the sun to the said line will be excluded from falling on the enlightened plane; and will cause a right-lined shadow on that part of it which would have been cut by the said plane of rays, had it not been intercepted. Thus the rays passing by the sun

(S, fig. 70) and the lamp iron, sign iron, &c. form planes, whose intersections with the planes of the ground, the cellar and the front of the house, mark on those planes the shadows of the said opaque lines.

THEOREM XXI.

CXI. The shadows of all lines which are parallel to each other, upon the same plane, or upon parallel planes, will be parallel.

DEMONSTRATION.

For the sun's rays are parallel (by § 106.) also the opaque lines are parallel by supposition: wherefore the planes of rays must be parallel (by E. 11. 15.) and their sections by the same plane, or by parallel planes, must be parallel (by E. 11. 16.) that is (by § 110.) their shadows, upon the same, or parallel planes, are parallel.

THEOREM XXII.

CXII. The shadow a b, (fig. 72.) which any right line (AB) casts upon a plane (W) to which it is parallel, will be parallel to the said line (AB).

DEMON-

DEMONSTRATION.

For since the line AB is parallel to the plane W , (which may represent a wall) a plane P may be passed by it, which shall be parallel to W , (*Vide* § 50.) then the line AB , and its shadow ab , (being the sections of these two planes, by a plane of rays ASB , passing through the sun, S ,) must be parallel (by E. 11. 16.)

COROLLARY I.

CXIII. *Hence, if the given plane and line are parallel to each other, and to the picture also, the representation of the shadow will be parallel to the representation of the line which causes it: for in this case, both the opaque line and its shadow, being parallel to each other and to the picture, must be represented by lines parallel to them and to each other (by § 53.)*

COROLLARY II.

CXIV. *But if the given plane and line are parallel to each other, but not parallel to the picture, the representations both of the given line and its shadow will tend to the same vanishing point (by § 17.)*

Q₃ THEOREM

THEOREM XXIII.

CXV. *If the rays proceeding from the sun (S. fig. 73.) are parallel to the picture (P) the shadows (b C, b C, &c.) of all lines (B C, B C,) that are parallel to the picture, on any oblique plane (G,) will be parallel to the entering line (E N) of that plane (G.)*

DEMONSTRATION.

For the rays B b, B b, &c. which come from the sun S, as well as the opaque lines B C, B C, &c. being parallel to the plane of the picture, by supposition, the planes of rays B C b, B C b, &c. must be parallel to the picture P. (by E. 11. 15.) but because these parallel planes B C b and P are cut by the plane G, the sections b C and E N are parallel (by E. 11. 16.) Q. E. D.

COROLLARY I.

CXVI. *Hence the shadows b C, b C, &c. of lines B C, B C, &c. which are parallel to the picture, caused by rays which are parallel to the picture upon any plane (G) which is oblique, must be represented by lines parallel to the vanishing line (V L) of that oblique plane G, for we have proved that the shadow b C, is parallel to the entering line E N, wherefore it must*

must be represented by a line parallel to the said *entering line* (by § 54.) or to the *vanishing line* (by § 6.)

T H E O R E M XXIV.

CXVII. *If the sun's rays are oblique to the picture, those rays which pass by any line (BC, BC, fig. 72.) that is parallel to the picture (P) will form an oblique plane (B b C) whose vanishing line (fU) will pass through the vanishing point (f) of the sun's rays, and will be parallel to the representation (bc) of that opaque line (BC).*

D E M O N S T R A T I O N.

Let I be the place of the eye, or point of sight, and I f a solar ray passing through it, and cutting the picture (P) in f; then is f the vanishing point of the sun's rays (by § 109.) Also through f, let fU be drawn on the picture P, parallel to bc (which is the representation of BC) then will fU be parallel to BC (by § 52 and E. 11. 9.) then because fI is parallel to Bb (by § 106.) and fU to BC (by construction) a plane passing by UIf is parallel to the plane of rays, B C b. (E. 11. 15.) And because it passes through the eye, it is the *vanishing plane* of the plane B C b; and its intersection, fU, with the picture P, will be the *vanishing line* of the said plane of rays, B C b. (by § 3.)

T H E O R E M XXV.

CXVIII. *The rays (B b, A a, &c. fig. 72 and 73.) which pass by any opaque line (A B, A B, &c.) that is inclined to the picture, (P) will form an oblique plane, (A B b a) whose vanishing line (S V) will be the representation of a ray of the sun, drawn through the vanishing point (V) of the given line. (Vide § 107 and 108.)*

D E M O N S T R A T I O N.

Suppose the sun's rays to be parallel to the picture, (P, fig. 73.) Through I, the place of the eye, suppose I V drawn parallel to the opaque line (A B) in question, and cutting the picture in V, then is V the *vanishing point* of the said opaque line. (by § 9.) On the picture, through V, suppose V f to be drawn parallel to the sun's rays B b, a A, &c. Then since V I and V f are respectively parallel to A b and B b, a plane, V I M f, passing by the former, will be parallel to the plane of rays, A a B b, passing by the latter; and because it passes through the eye, (I) will be its *vanishing plane*, (by § 3); and V f will be its *vanishing line*; but the same line, V f, is the *representation* of a ray drawn through (V) the vanishing point of the opaque line, A B. (by § 107.) Q E D.

C A S E

C A S E II.

If the sun's rays are oblique to the picture (P. fig. 72.) the demonstration will be nearly the same. For if $I f$ is a solar ray passing through the eye, and cutting the picture in f ; also if $I V$ is parallel to the opaque line, $A B$, then is V the vanishing point of $A B$, (by § 3.) and f the vanishing point of the solar rays. (by § 108.) Again, because $I V$ and $I f$ are respectively parallel to $A B$ and $B b$, the plane, $V I f$ is the vanishing plane of $A B a b$ (by E. 11. 9. and § 3.) and $S V$ is its vanishing line; but $S V$ is the representation of a solar ray drawn through the point V . (by § 108.) Q. E. D.

T H E O R E M XXVI.

CXIX. *If the plane of rays, ($A B a b$, fig. 72 and 73.) which pass by any opaque line, (AB) is oblique to the picture, (P) the point (V) where its vanishing line ($f V$) cuts the vanishing line ($V L$) of any oblique plane, (G) will be the vanishing point of the shadow ($a b$) of the said opaque line ($A B$) and all its parallels, upon the said oblique plane, (G) or any other plane parallel to it.*

D E M O N -

DEMONSTRATION.

For the shadow, $a b$, (fig. 72 and 73.) of any line, $A B$, upon any plane, G , is the intersection of the plane of rays, $A B a b$, passing by the said line, $A B$, with the said plane G , (by § 110) consequently the shadow, $a b$, being in the plane of rays, $A B a b$, its vanishing point must be somewhere in the vanishing line ($f V$) of the plane of rays. (by § 87.) Also the said shadow ($a b$) being in the oblique plane, G , its vanishing point must be somewhere in the vanishing line, $V L$ of the plane G : wherefore seeing it is in both these vanishing lines $V L$ and $V f$, it must be in their intersection (V .) The rest of the proposition is manifest from § 111 and 17.

L E M M A.

CXX. *If a plane of rays ($A B a b$, fig. 73.) passing by an opaque line ($A B$) cuts a plane ($B c b$) that is parallel to the sun's rays, the section ($b B$) will be a ray tending to the sun.*

In this case, since both the planes, $A B a b$, and $B C b$ pass through the sun, (§ 106) their intersection, $B b$, must pass through the sun.

C O R O L -

C O R O L L A R Y.

CXXI. Hence the shadow (B b fig. 73.) of any right line whatsoever (as A B) upon a plane (B C b) parallel to the sun's rays, will be a ray passing through the sun (*Vide* § 110) and its section (B) with the given plane: (B C b) also such shadow (B b) must be of infinite extent, because the ray (A a) passing through the other end (A) of the opaque line, will never cut the given plane (B C d.)

N. B. Properly speaking, a plane that is parallel to the sun's rays, cannot receive a shadow; because the sun's rays do not cut it, but pass by it, (*Vide* § 105.) nor can such a plane be said to be enlightened by the sun, because none of its rays fall on it; I conceive therefore, that what Ozanam and some other writers say of an infinite shadow is rather absurd. It must be owned, however, that if the plane is *very near* parallel to the sun's rays, a shadow may be formed on it, which will very nearly tend to the sun and be of great extent.

The foregoing theorems and observations will enable us to find the representation of the shadow of a right line upon any plane whatsoever; but to render every thing perfectly intelligible, I shall give a praxis of all that has been said on this subject.

E X A M-

E X A M P L E.

When the sun's rays are parallel to the picture.

In fig. 70. we suppose the sun's rays parallel to the picture; wherefore their representations, which are parallel to each other, may also be parallel to any assumed line, as R S. (*Vide* § 107).

HL is the horizontal line,

C is the center of the picture,

VI is the vanishing line of the front of the cellar,

Um is that of the cellar door which is open,

CU is the vanishing line of the oblique front of the house.

These we suppose were determined when the representations of these objects were drawn on the picture.

The lamp-post is parallel to the picture, wherefore its shadow *upon the ground* is parallel to HL; (by § 116.) *upon the cellar door*, its shadow is parallel to Um, and upon the oblique front of the house its shadow is parallel to CU, where it is finally determined by a ray drawn through the top of the lamp. (*Vide* § 107).

The ladder is oblique to the picture; its vanishing point is A. AL is the representation of a ray drawn through A, (*Vide* § 107.)
cut-

cutting HL in L , Vl in a , Um in b , and CU in c . Then this line, AL , is the vanishing line of the plane of rays passing by the ladder: (*Vide* Art. 118, Case 1.) Wherefore, by § 119, the shadow of the ladder on the ground tends to L ; on the front of the house it tends to c . On the parallel front, M , of the cellar, it is parallel to RS . (*Vide* Art. 107 and 121.) On the slant front of the cellar it tends to a . On the cellar door it tends to b . On the oblique front of the house it again tends to c ; and is finally terminated by a ray drawn from the top of the ladder, parallel to RS . (*Vide* § 107.) I have here traced the shadow of the ladder, progressively from its bottom to its top. But it must be remembered, that the shadow on the lower part of the front of the house and on the parallel front (M) of the cellar, will be hid by the shadow of the cellar door; which therefore should have been found first: And when we had traced the shadow of the ladder along the ground to (d) the bottom of the house, we might have drawn dc to the vanishing point (c) of the front, and cutting it by a ray, parallel to RS ; (*Vide* § 107.) the shadow of the top would be obtained the same as before; from whence the shadow on the top of the cellar and the top of the cellar door might have been drawn backwards. This is only mentioned as a hint that the learner, by using
his

his sagacity, will sometimes save trouble.—
 Let us now trace the shadow of the cellar
 door—Its edges are parallel to the picture,
 and its ends tend to p (which is the center
 of its vanishing line, U m.) Through p,
 draw p n, parallel to R S, and it is the
 vanishing line of the plane of rays passing by
 the ends of the door, and cuts U C in o and
 H L in n. (*Vide* § 118 and 119.) Wherefore
 from the lower corner of the cellar, r, draw
 a line tending to n, and cut it in q, by a
 line drawn through the nearest lower corner
 of the cellar door, parallel to R S. (*Vide* §
 107.) And by this means I get the shadow,
 q r, of the lower edge, which falls wholly
 on the ground. The shadow of the nearest
 parallel edge, on the ground must pass thro'
 q, parallel to H L, (by § 116.) and on
 the door of the house it is parallel to U C;
 where it is terminated by a line drawn thro'
 the upper corner of the cellar door, parallel
 to R S. (*Vide* § 107.) From this termina-
 tion we trace the shadow of the upper end of
 the cellar door, which on the door of the
 house tends to o, (by § 119.) and on the
 recess, from the front of the house to the
 door, (which recess is parallel to the picture,
 and consequently to the sun's rays) the sha-
 dow (if any) will be parallel to R S (by §
 121.) The shadow of the cellar door is now
 finished, for the rest becomes hid by the door
 itself. I shall next trace out the shadow of
 the

the pent-house over the door; the shadow of its parallel edges, on the front of the house, are parallel to UC (by § 115,) and are terminated by lines drawn through the corners of the pent-house, parallel to RS (*Vide* § 107.) Also the shadow of the oblique edge of the pent-house, on the front of the house, and the door will tend to C (by § 114.) Also its shadow on the recess of the door (if any) will be parallel to RS (*Vide* § 107. and 121.) The sign, with the iron on which it hangs, and the upper iron which supports it, are all parallel to the picture, and consequently in the same plane with the sun, wherefore, it is easy to conceive, the shadow of all these will be one right line, parallel to UC , (by § 115.) and pass through the point where the sign iron cuts the front of the house; and the point q in this line, where it is cut by a line parallel to RS , and passing through the corner of the sign Q will be the shadow of that corner (by § 107.) Also a parallel to RS , through the point D (where the three irons which support the sign, meet) gives the d for the shadow of that point; wherefore, Fd is the shadow of the oblique iron (FD) on the front of the house; but this shadow crosses the middle window, and therefore the learner must recollect that on the farther recess of the window, the shadow will be parallel to RS (by § 107. and 121.) and on the plane of the window it will tend

tend to the same vanishing point as F D does (by § 111. and § 17).

We have now done with the oblique front, and must proceed to the parallel front, on which, properly speaking, no shadows can fall, (*Vide* note, page 235.) because this front is in the plane of the sun's rays; therefore, to be consistent we must suppose the sun not to be quite parallel to the picture but to be so posited as to cast on the parallel front such shadows as we observe the sun cast on a level plane, when it has been risen about five or ten minutes, which are all parallel to each other and of a very great length. We have, therefore, only to observe concerning the shadows represented on this plane, that they are all parallel to each other and to R S. *Vide* § 52, 107, and 121.

E X A M P L E II.

When the sun's rays are oblique to the picture.

In fig. 71, the sun is supposed behind the spectator, (as in fig. 72.) and S is assumed any where below the horizontal line, H L, for the vanishing point of the sun's rays. (*Vide* § 108 and 109.)

The same objects are drawn in this scheme as in the last, and I shall investigate their shadows in the same order.

H L

HL is the horizontal line,

VL is the vanishing line of the cellar,

uH is the vanishing line of the cellar door, which is now opened on the farthest side of the cellar; And

UC is the vanishing line of the oblique side of the house.

The lamp-post is parallel to the picture, wherefore SL, drawn through S, parallel to it, is the vanishing line of the plane of rays passing by it: (*Vide* § 117.) this cuts HL in L, VL in V, and uH in u: wherefore, on the ground, the shadow tends to L; and it should be continued by an occult line till it cuts that part of the bottom of the house which is hid by the cellar, through which point a line drawn parallel to the lamp-post will represent the shadow on the front of the house, (by § 113) where it may be terminated by drawing a line through the top of the lamp and the point S. (*Vide* § 108.) To determine its shadow on the cellar door, we must take notice that the corner (a) of the cellar door touches the front of the house, and that uH cuts UC in b, wherefore ab is the line in which the cellar door would cut the front of the house if it was continued to it. But ab cuts the shadow of the lamp-iron in c, then a line drawn through c and u (*Vide* § 119.) will be the shadow of the lamp-iron on the cellar door.

R

The

The ladder is oblique to the picture, and A, is its vanishing point; wherefore AS is the vanishing line of the planes of rays passing by its sides. (*Vide* § 118.) But this line (AS) cuts uH in d, V1 in e, HL in f, and UC in g; wherefore, (by § 119.) the shadow of the ladder on the ground tends to f (and let it be continued by occult lines till it cuts the bottom of the front of the house) on the front of the house, it tends to g, on the parallel front of the cellar the shadow is parallel to AS (*Vide* § 54 and 6.) on the cellar door which is shut, the shadow of the ladder tends to e, and on that which is open, it tends to d.

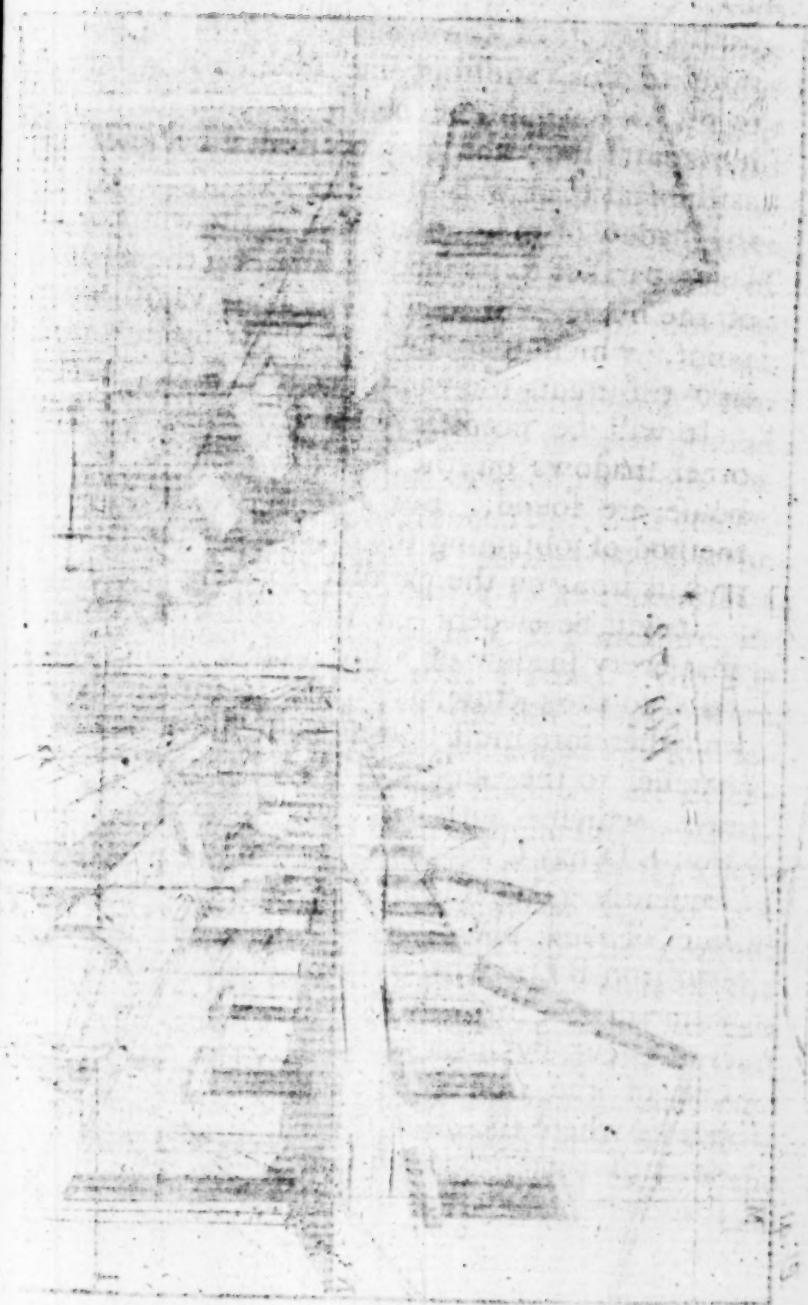
The shadows of the sign and its irons fall wholly on the oblique front of the house and the recesses of the windows. The horizontal iron kl is parallel to the picture, wherefore, SG drawn parallel to the said iron, is the vanishing line of the plane of rays passing by it. (*Vide* § 117.) But SG cuts UC in h, which is the vanishing point of the shadow of the horizontal iron (kl) and the top and bottom of the sign on the front of the house: wherefore I draw kh, and cut it in o by a line drawn thro' l to S. Through o, I draw a short line perpendicular to the horizontal line, and cut it by a line drawn to S from the point where the other two irons (ml and nl) unite, and so obtain the shadow

dow of that point, to which I draw lines from m and n, and they will be the shadows of the said irons on the front of the house. To obtain the shadow of the sign, I produce its sides, till they cut the iron k l in q and r: through these points lines drawn to S give the shadow of those points by cutting k h: lines drawn through these points of shadow, parallel to the perpendicular edges of the sign, will be their shadow (*Vide* § 113.) whose limits will be determined by drawing lines to S from the lower corners of the sign. The shadows of the sign-irons are broke by the recesses of the windows, concerning which we may observe, that because k l and n l are parallel to the picture, their shadows on the parallel recess of the farthest window will be represented by lines parallel to them: (*Vide* § 113.) also, on the oblique plane of the window they will be parallel to their shadow on the front of the house already found, and consequently will tend to the same vanishing points, which are found by producing the said shadows till they cut the vanishing line U C. The iron m l is parallel to the horizon and oblique to the picture, so that if produced till it cut the horizontal line, we should obtain its vanishing point (by § 88.) to which a line drawn from S would be the vanishing line of the plane of rays passing by m l; (by § 118.) then the shadow on the

parallel recess of the windows would be parallel to this vanishing line (which it is easy to perceive would be nearly parallel to the horizontal line, and may be drawn by guess, as the said shadow is of short extent). Also the shadow of $m l$ on the plane of the window, being parallel to its shadow, $m o$, on the front of the house, will tend to the same vanishing point, which may be found by producing $m o$ till it cuts the vanishing line $U C$.

It will be needless to describe how the other shadows on the oblique front of the house are found. But I shall describe the method of obtaining the shadows of the lamp and its irons on the parallel front.

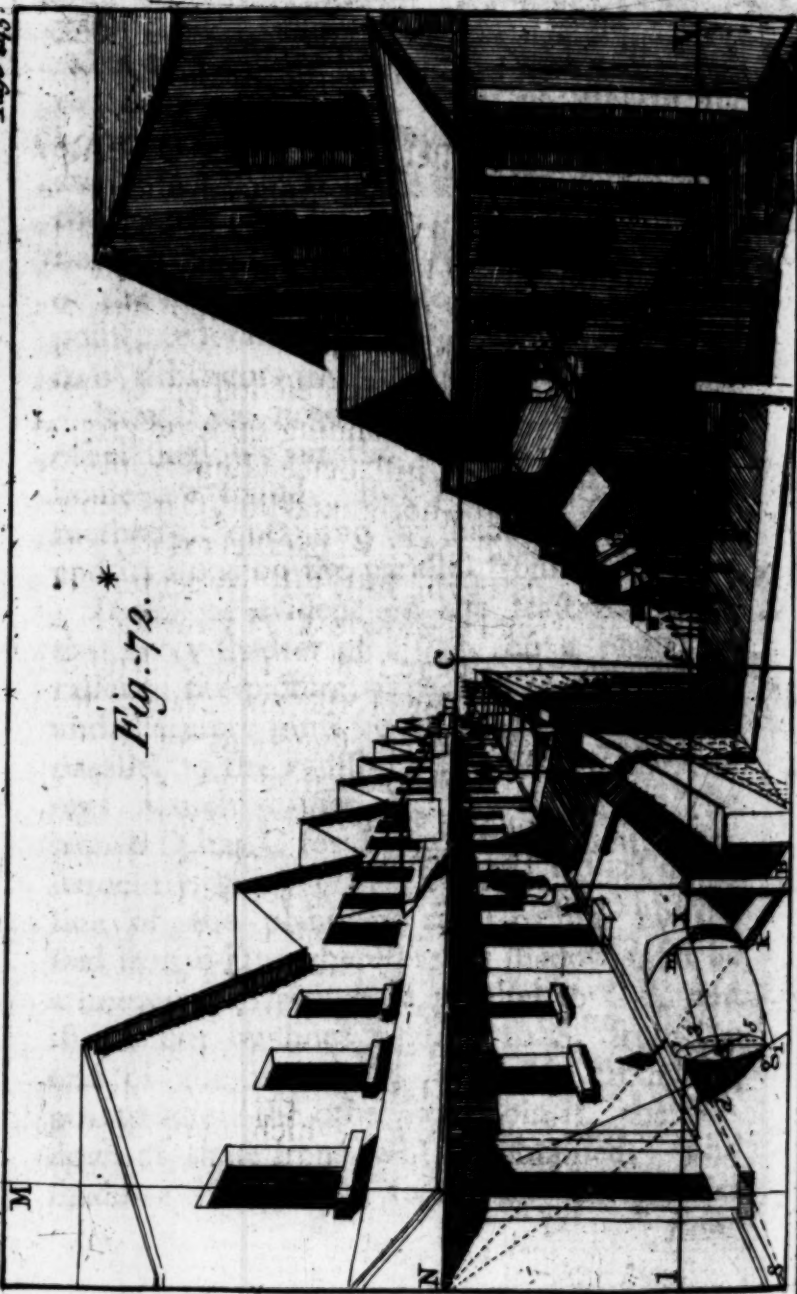
It will be evident on the least reflection, that every shadow of a line, on a plane parallel to the picture, is parallel to the picture, and therefore must be represented by a line parallel to the vanishing line of the plane of rays which causes it. (by § 44.) The iron $B D$ has C for its vanishing point, consequently (by § 118.) $S C$ is the vanishing line of the plane of rays passing by the said iron $B D$; wherefore its shadow must be a line drawn through D parallel to $C S$, and if it is cut by lines tending to S , from the end of the iron $B D$, and also from the points where the other irons join it, the shadows of those irons will be obtained. The shadows being thus found on the parallel front



Page 245.

*
Fig. 72.

Pl. 41.



front of the house, they may easily be broke into the recess of the window, remembering that on the oblique recess the shadow of the iron BD tends to C (by Art. 114.) because that recess is parallel to the iron BD. Also the shadows of the other irons on the plane of the window will be parallel to those parts of their shadow which fall on the front of the house. Every thing else is too obvious to need farther explanation.

In fig. 72. is the representation of two rows of houses, one standing on level ground and the other on a hill descending from the picture. H L is the horizontal line, C its center, and H C is its distance: V l parallel to H L is the vanishing line of the hill, f is its center, H f its distance, and the angle C H f measures its declivity. (*Vide* page 178, § 102.) Also S is the vanishing point of the sun's rays: from whence, how the representations of the several objects, with their shadows, are determined, is proposed as an exercise for the learner's skill. But it may be proper to say a few words concerning the shadow of the post on the cask, and of the cask on the ground. L M is the vanishing line of the ends of the cask, and H is the vanishing point of its axis (A X). From A the center of the nearest end of the cask, I draw a perpendicular, cutting the ground in g, (which must be a little below the circular chime 1354, because the

cask rests on its bilge) through g draw gL , which is the section of the head of the cask with the ground. Also through B (the bottom of the post) draw BN , which represents the shadow of the post on the ground: through any point (3) in the circumference of the head of the cask, draw $3i$ parallel to Ag , and cutting gL in i ; through i , draw iH cutting BN in k ; through k , draw km parallel to $3i$. Also through 3 , draw a line tending to H , or a little above or below it, according as is thought necessary on account of the bilge of the cask; and the point m , where this line cuts km , will be one point through which the shadow of the post must pass. For it is evident that $ikm3$ is the representation of a plane perpendicular to the ground, and whose upper side $3m$ very nearly agrees with the surface of the cask, and that km is the shadow of the post on that plane: so that m is situated in the shadow of the post, and also on the surface of the cask. More points may be found by repeating the operation; but when two or three points are found, the curve, representing the shadow, may be drawn by hand, as great exactness would be needless. To find the shadow of any point in the head of the cask on the ground, as the point 1 , through its seat g , draw gN , and draw $1S$ cutting it in o . In like manner may the shadows of other points be

be found, and a curve drawn through them marks the boundary of the shadow: but to be very exact in these kind of shadows would be as useless as tedious: for when the mind is informed by science, the hand may often be permitted to supersede the ruler and compasses.

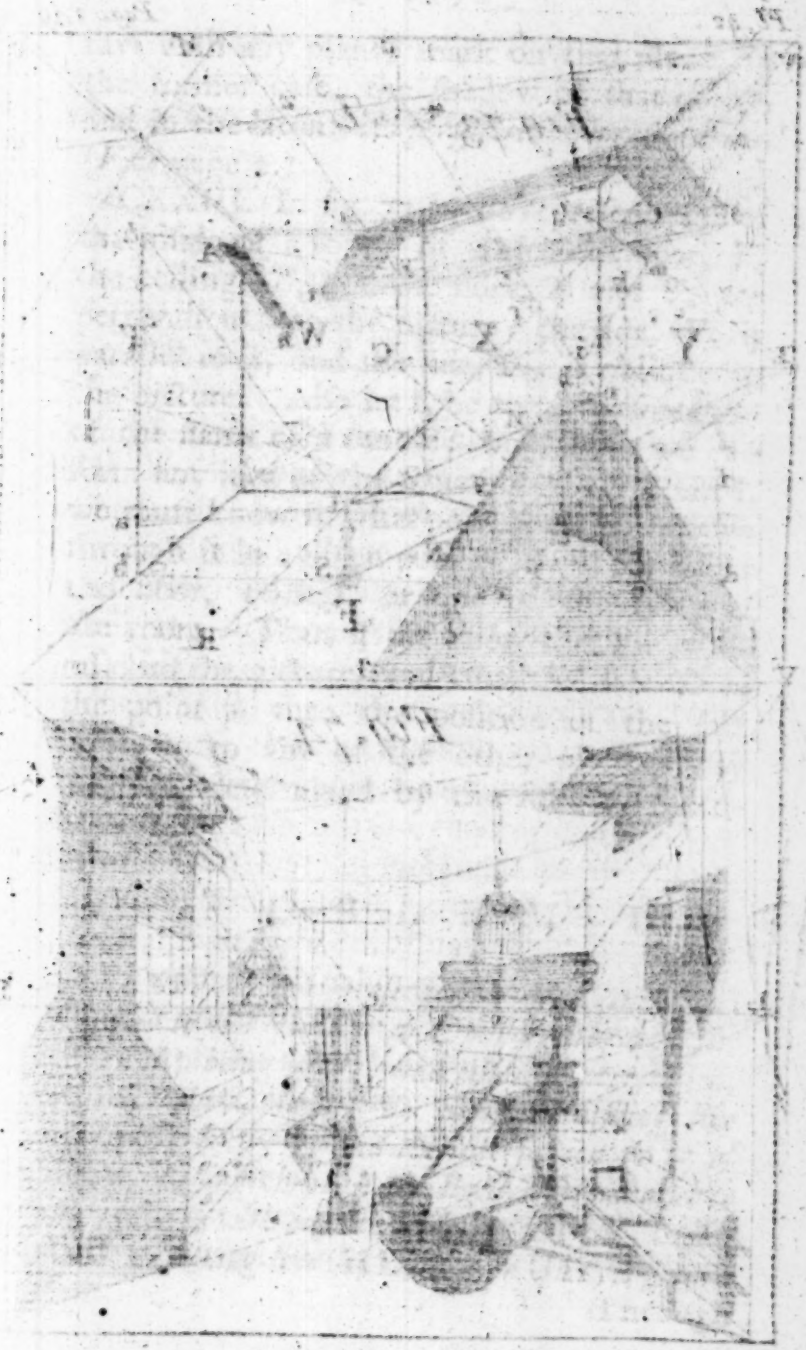


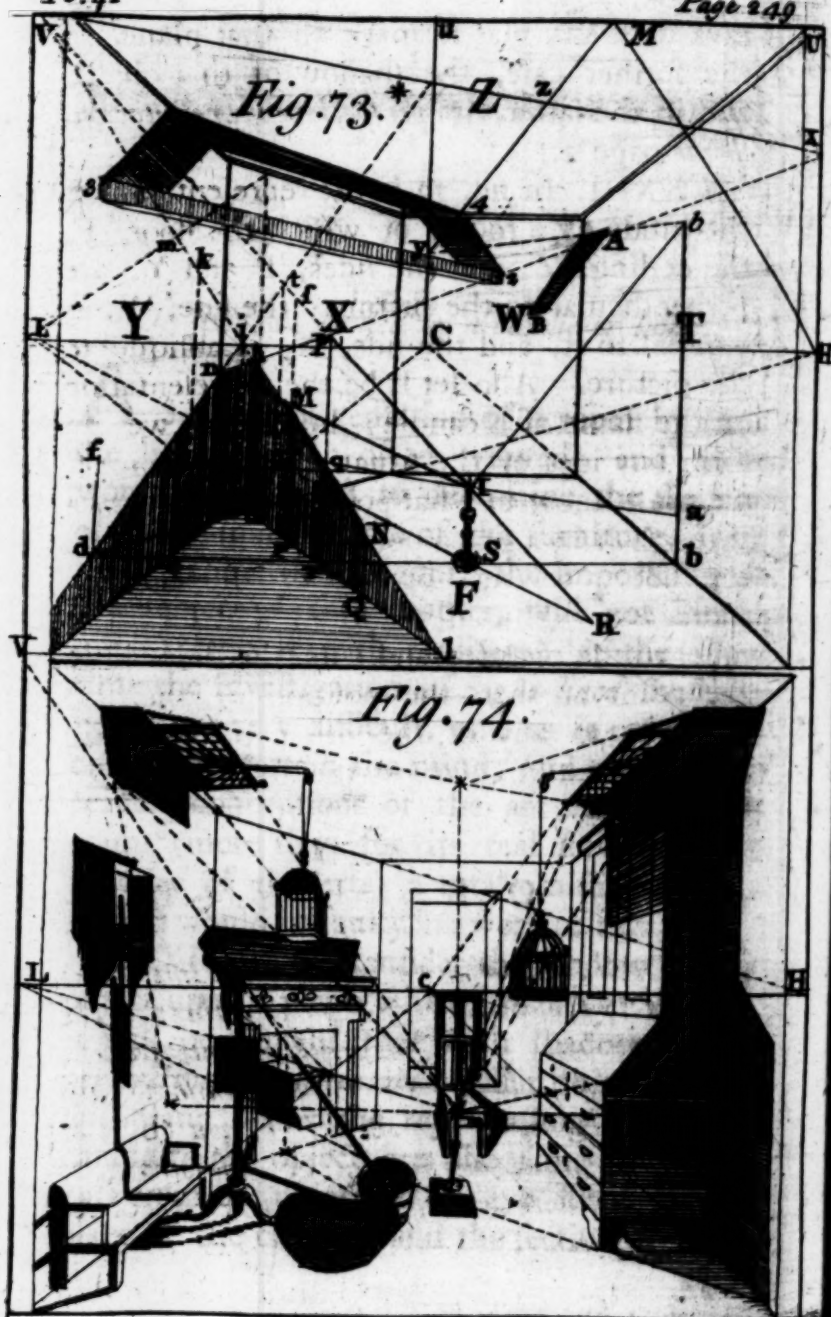
C H A P. XI.

Of shadows caused by a candle or lamp.

AS the representation of a room by candle light is not often wanted, and as it would be needless to determine the shadow of every small article of the furniture, with strict geometrical rigour, the importance of the subject of this chapter, will not intitle me to treat it with prolixity: at the same time the investigation of *right-lined* shadows being not very difficult, and as it may exercise and entertain the mind, and enlarge the readers conceptions of the art; on this account (more than for its real service in the practice of the arts) a total omission of this subject would be faulty in a work of this kind.

CXXII. If we consider the shadow of any object upon a plane when caused by a candle, it will be evident that such shadow is the *perspective representation* of that object on the said plane. For the rays passing from the candle to the object are the same as would pass from the object to an eye fixed in the place of the candle; and the section of these rays,





rays with any plane, mark on that plane, in the former case, the shadow of that object, and in the latter, its *perspective representation*. (*Vide page 2.*)

CXXIII. In fig. 73 is the representation of the inside of a room, of which the floor, F, the ceiling, Z, and the sides, T and Y, are perpendicular to the picture; the side, W, is parallel to it, and the side, X, is oblique to the picture. Also let I, be the representation of the flame of a candle: but before we can form any idea of the situation of the point I, we must know in what point some line drawn through it in a given position, will cut either the floor, ceiling, or one of the sides of the room. Thus if we know that Ia is parallel to the picture, and cuts the side T in the point a, then the position of the said point I, to any of the other planes, may easily be determined by the following problem.

P R O B L E M I.

CXXIV. *The representation of a line (Ia) parallel to the picture and passing through the light, (I), being given, and also its intersection, a, with some plane (T) whose vanishing line (u C) is given; to find the intersection (b S) of a plane passing through the light (I) parallel to the picture, with any other plane (as F, Y, or X) whose vanishing line (HL, u C, or UH) is given.*
Through

Through a , I draw a line ($a b$) parallel to the vanishing line of the given plane, T , and it cuts the ground in b . Through b , $b S$, drawn parallel to the vanishing line of the ground, cuts the plane, Y , in c ; and the plane, X , continued, in d : through c and d , lines ($c e$ and $d f$) drawn parallel to the vanishing lines of the planes X and Y will be the sections of a plane passing thro' the candle, and parallel to the picture, with the sides of the room X and Y , and $b c$ is its section with the ground plane F , all which is evident from § 54, and E. 11. 15.

P R O B L E M II.

CXXV. *Having the representation (MN, fig. 73) of a line given; and the points (P and Q) where two lines (NQ, and MP) drawn from it, parallel to each other and to the picture, cut a plane, F, whose vanishing line (HL) is given; to find the vanishing point (V) of that line, and its intersection with any other plane (X) whose vanishing line (UH) is given.*

Through P and Q draw a line, and produce it till it cuts the vanishing line HL in L : through L draw LV parallel to NQ , or MP ; also produce NM till it cuts VL in V ; then is V the vanishing point of NM . For the vanishing line of a plane passing by QP , must pass through L (by § 87.) also the vanishing line of a plane passing by NQ must

must be parallel to it. (by § 54.) wherefore VL is the vanishing line of the plane which passes by NQ and QL ; but PN and MN are also in the same plane (by E. 11. 7.) wherefore the point V , where MN produced cuts VL , is its *vanishing point* (by § 88.) The former part of our problems being solved, the latter is easily effected. QL cuts the plane X in g : now in this example the vanishing line VL is parallel to the vanishing line of the plane X ; wherefore the planes represented by X and $VNQL$ intersect each other in a line parallel to the picture. (*Vide* § 102) and therefore the representation of such intersection must be parallel to (VL , or, to UH) the vanishing line of the plane X (by § 54.) wherefore gh drawn parallel to VL , is the section of the plane $VLNQ$ with the plane X ; and the point h , in which it cuts NM produced, must be the point in which the said MN cuts the plane X : in like manner we find that MN , if produced, would cut the plane Y , produced behind X , in i : also it would cut the plane W produced behind X in the point k , and it would cut the plane F in the point l , and the ceiling Z produced behind Y , in m .

C O R O L L A R Y.

CXXVI. *It is self-evident that the shadow of any line (MN) on a plane (X) must pass through that point*

point (h) where the said line, (MN) when produced, cuts the said plane.

For this reason, the shadow of the line, MN, on the plane Y, passes through i; and on the plane F, it passes through l.

P R O B L E M III.

CXXVII. *The representation (MN, fig. 73.) of a line being given, thro' the flame of the candle (I) to draw the representation (VI) of a line parallel to it. 2d. To find the intersection (p) of this line (VI) with any plane, X, (whose vanishing line (HU) or position is known). And, 3d. To prove that the shadow of any line (MN) and of all lines parallel to it on any plane (X) tends to that point (p) where it is cut by a parallel (VI) drawn through the flame of the candle or lamp.*

If the vanishing point V of the given line MN is known, the first part of our problem is solved by problem 8. of chapter 9. If this vanishing point is not known, it must be found by such data as were given in the last problem. For the second part, I draw the representation of a plane ace (by § 124) parallel to the picture and passing through the light; and on this plane draw IS parallel to UH, cutting the floor F, in S; also through V, VL is drawn parallel to UH, and cutting HL in L. Lastly, I draw SL, and it cuts the plane X in o, and
drawing

drawing op parallel to UH or to VL , it cuts IV in p , which, therefore, is the point where IV cuts the plane X , as is evident from the last problem. In like manner I find IV , if produced, cuts the ground in R : also it cuts the plane Y , produced behind X in t ; and the plane W , produced behind X , is cut by the said IV in f . I am now, in the third place, to prove that the shadow of MN (or any line parallel to it) upon any plane (as X) will pass through the point (p) where the said plane is cut by a line (VI) drawn through I , parallel to MN ; and this is evident if we reflect on what was said in § 122, for if we consider the plane, (X) on which the shadow falls, as the picture plane, and the eye to be placed at I (instead of the light) then the line VI will be the distance of the given line MN , and the point (p) where it cuts the plane X , will be its vanishing point, (by § 9.) but the representations (or, which is the same thing, the shadows) of all lines parallel to MN , tend to the same vanishing point. (by § 17.) wherefore the shadow of MN , and of all its parallels, on the plane X , tends to the point p . Q. E. D.

P R O B L E M IV.

CXXVIII. *Having given the representation of a line, also its Intersection with a plane, whose vanishing line is given; to find the vanishing line of the plane of shadow which passes by that line.*

C A S E

C A S E I.

When the given line is parallel to the picture.

Let AB (fig. 73.) be the representation of a line parallel to the picture, cutting the plane T , (whose vanishing line is $u C$) in the point A . Through I , draw (by prob 1. § 124) the representation of a plane parallel to the picture, and find the line $a b$, in which it cuts the given plane T ; also through I , draw $I b$ parallel to AB , and cutting $a b$ in b : draw $b A$, and produce it till it cuts $u C$ in y . Lastly, draw $y z$ parallel to AB , and it is the vanishing line of the plane of shadow passing by AB . For it is evident, from the latter part of the last problem, that the plane of shadow passing by AB , cuts the plane T in the line $b A$, of which y is the vanishing point, (by § 88) also the vanishing line of any plane passing by AB , must be parallel to it (by § 54) wherefore $y z$ is the vanishing line sought (by § 87.)

C A S E II.

When the given line is not parallel to the picture.

Let $l M$ (fig. 73) be the given line, whose vanishing point is V ; and let it cut the plane F (whose vanishing line is HL) in the point l . Thro' V , draw VL (in any direction at pleasure) cutting HL (in any point, as) in L ; and draw
 $l L$;

I L; conceive V I L to be the representation of a plane, whose vanishing line is V L, and (by prob. 1, § 124) find its section q r, with a plane passing through the light I, and parallel to the picture. Lastly draw I q, and V x parallel to it, which will be the vanishing line sought. For if I V is drawn, it is plain I V q is the representation of an infinite parallelogram passing through I, and the given line l M, and cutting the parallel plane I S r q, in the line I q, which therefore is parallel to the picture; consequently the vanishing line of the plane I V q, must pass through V (by § 87) and must be parallel to I q (by § 54)

C O R O L L A R Y.

CXXIX. *The shadow h n of a line (MN) on a plane (X) passes through that point (x) where the vanishing Line (V x) of its plane of shadow (I q V.) cuts the vanishing line (U H) of the said plane (X) Vide § 119, page 233.*

C O R O L L A R Y II.

CXXX. *The shadow of any line on a plane parallel to the picture, is parallel to the vanishing line of its plane of shadow (Vide § 54.)*

Thus 23 (fig. 73) is a line parallel to the ground or ceiling, and H M is the vanishing line of the plane of shadow passing by it and the candle I, wherefore its shadow 24, on the plane W, is parallel to H M.

A. THE-

A T H E O R E M.

CXXXI. *The shadow of any line (caused either by the sun or a candle) on a plane to which it is parallel, will be parallel to that line.*

This demonstration is the same as in § 112; and the same conclusions may be drawn as are mentioned in § 113 and 114.

A N A X I O M.

CXXXII. *The shadow of any point (M fig. 73) is situated in a line (I M) passing through the light (I.) and the said point.*

The learner is now furnished with every thing that can conduce to the finding of right lined shadows; and his own sagacity will point out how he may obtain points for the formation of those curves which are generated by the shadows of curves on planes, or lines on curved surfaces (*Vide* page 245.) As a summary of the whole, I have added, in fig. 74, the representation of a room, with the shadows of the several objects therein; and it may afford the mind an agreeable amusement to observe how the various methods, which will spontaneously result from contemplating the foregoing problems, concur in producing the same effects.

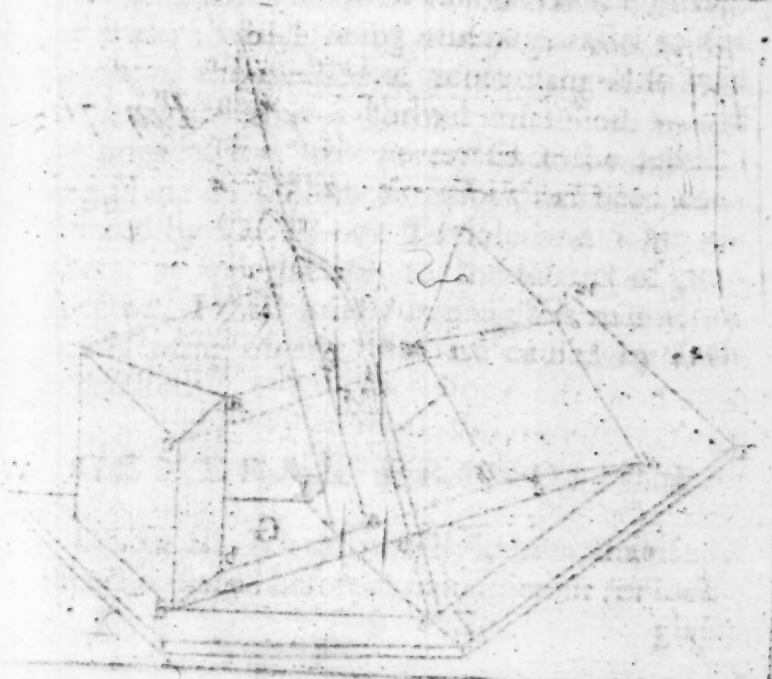
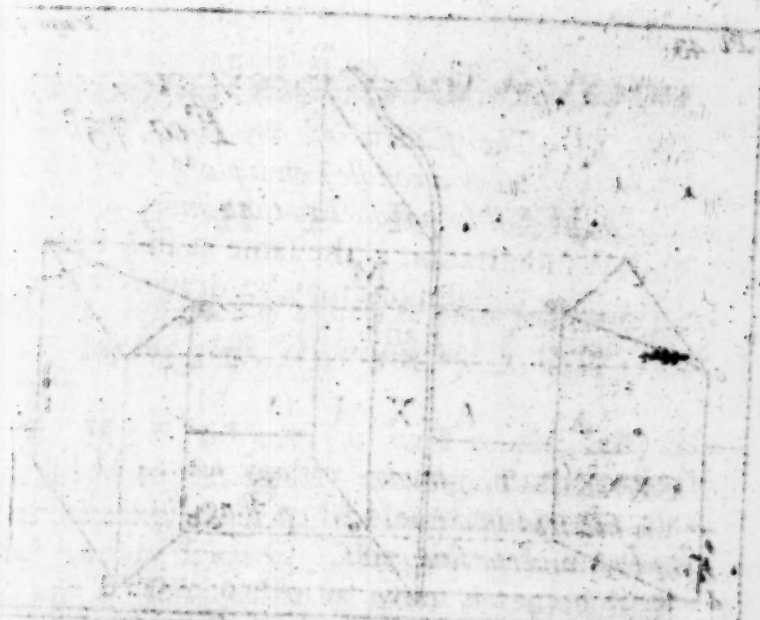


Fig. 75.

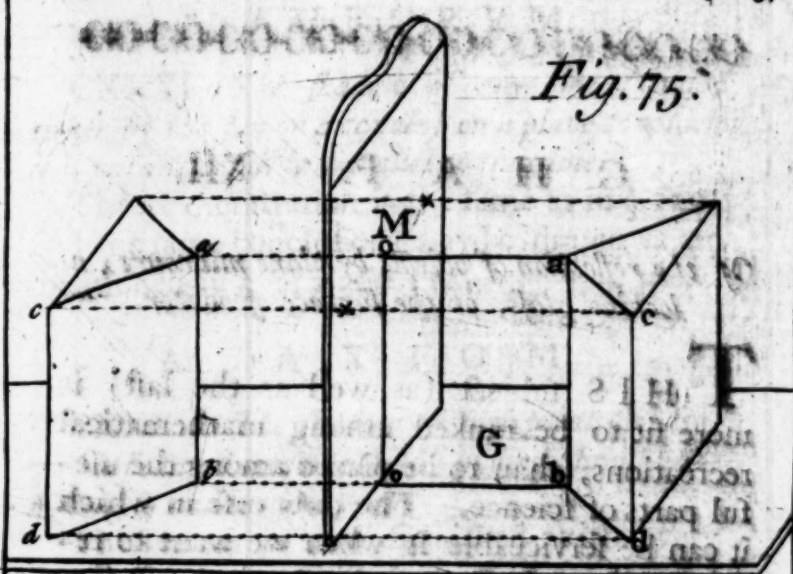
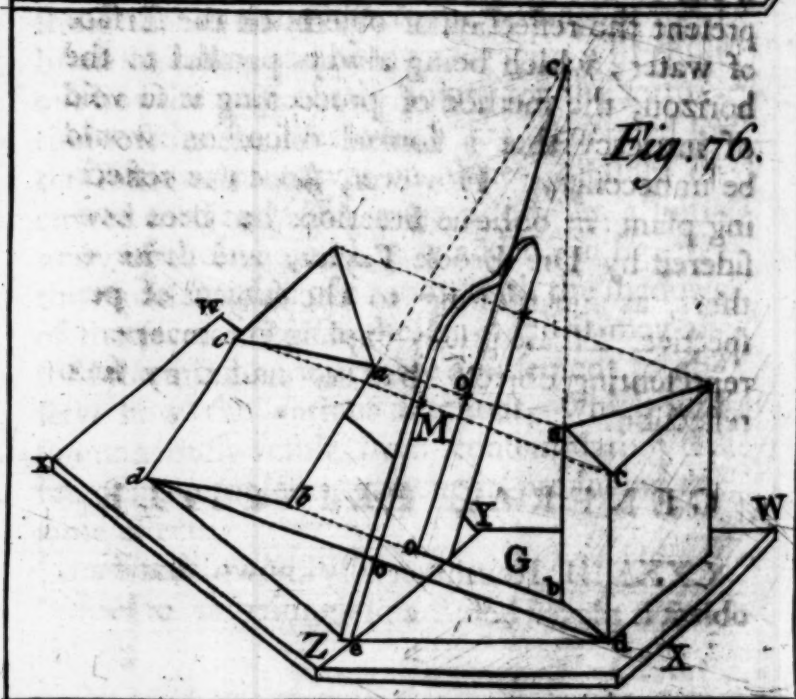
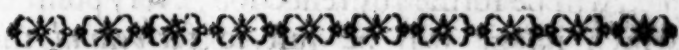


Fig. 76.





C H A P. XII.

Of the reflection of objects by plane mirrors; as looking glass, or the surface of water.

THIS subject (as well as the last) is more fit to be ranked among mathematical recreations, than to be placed among the useful parts of science. The only case in which it can be serviceable is when we want to represent the reflection of objects on the surface of water; which being always parallel to the horizon, the method of proceeding is so void of intricacy that a formal discussion would be unnecessary. However, since the reflecting plane in oblique situations has been considered by Dr. Brook Taylor, and some others, as appertaining to the subject of perspective, I shall briefly explain the manner of representing objects that are caused by such reflection.

GENERAL PRINCIPLE.

CXXXIII. It is universally known, that if any object is placed before a plain mirror (or looking glass)

ing glass) an image, exactly similar to that object, will be formed by reflection; every point of which image will be just so far *behind* the glass as its corresponding point in the real *original object* is before the glass. Thus let a (fig. 75 and 76.) be the corner of a prism standing on a plane, G , from which let ao be drawn perpendicular to the plane of the glass, M , and cutting it in o , then if ao is continued, and oa is made equal to ao , the point a will be the point where the image of a is formed.

From hence we may conclude, that when the real object and the mirror are fixed, the image formed by reflection will appear as if actually existing in a fixed position; and that if we would draw a *perspective representation* of the image thus formed, we have nothing more to do than to consider this *reflected image* as a real object, of the same dimensions and shape with the original object which causes it, and placed in the situation which the above-mentioned property of a *plane-mirror* will throw it in.

THE OPTIC. XXVII.

CXXXIV. The reflection (ab , fig. 75 and 76) of any line ($a b$) is situated in a plane ($a a b b$) passing by that line perpendicular to the mirror (M). For the reflection of any point, a , in the line ab is in a line aa , perpendicular to

to the mirror M , (by § 133.) now therefore
the plane $a b$ is perpendicular to the mirror,
(by E. 11. def. 8.) Now again which
line is the reflecting point in the
T H E O R E M XXVIII.

CXXXV. The angle $(b c o, \text{fig. 76})$ which any
real line $(a b)$ makes with the face of the mirror
(M) is equal to the angle $(o c b)$ which its re-
flection $(a b)$ makes with the back of the mir-
ror (M).

Let $a b$ be produced till it cuts the mir-
ror M in the point o ; and from the extre-
mity b , draw $b o$ perpendicular to the mir-
ror, and cutting it in o ; then if $o c$ is drawn,
the angle $b c o$ measures the inclination of the
line $b c$ to the mirror M . (by E. 11. def. 8.)
Let $b o$ be produced, and make $b d = b o$,
then is b the reflection of b , and $b c$ will
be the reflection of $b c$ (by § 133.) and because
in the two triangles $c o b$ and $c o b$, the two
sides $c o$ and $b o$ and the included right angle
 $c o b$, are respectively equal to the two sides
 $c o$ and $b o$, and the included right angle
 $c o b$; therefore the angle $b c o$ is equal to
the angle $b c o$. (by E. 1. 4.) Q. E. D.

T H E O R E M XXIX.

CXXXVI. The angle $(d e o, \text{fig. 76.})$ which
any real plane $(W X Y Z)$ makes with the face
of the mirror (M) is equal to the angle $(o e d)$

which its reflection ($YZwx$) makes with the back of the mirror.

For from any point (e) in the intersection (YZ) of the real plane with the mirror, draw de on the real plane, and eo on the mirror, each perpendicular to YZ ; then the angle deo is the inclination of the plane $WXYZ$ to the face of the mirror M (by E. 11. def. 6.) and the plane (ded) passing by these lines (de and eo) will be perpendicular to YZ , (by E. 11. 4.) and, consequently, to the mirror M , and to the plane $WXYZ$. (by E. 11. 19.) Wherefore, on this perpendicular plane (ded) draw do perpendicular to eo , and make $od = do$; then will ed be the reflection of de . And since the real plane ($WXYZ$) passes by de and YZ , its reflection ($YZwx$) must pass by ed and YZ , and, consequently, is measured by the angle oed which is equal deo .

C O R O L L A R Y I.

CXXXVII. Hence the triangle bco is in all respects equal and similar to its reflection bco : and the trapezium $booa$ is equal and similar to its reflection $booa$. And the reflection of any plane figure is equal and similar to the real plane figure which causes it, and posited in the same situation with respect to the back of the glass, as the said real figure is to the face of the glass.

C O R O L

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1. The first figure is a square with side length 1. The second figure is a square with side length 2. The third figure is a square with side length 3. The fourth figure is a square with side length 4. The fifth figure is a square with side length 5. The sixth figure is a square with side length 6. The seventh figure is a square with side length 7. The eighth figure is a square with side length 8. The ninth figure is a square with side length 9. The tenth figure is a square with side length 10.

COLA

E X A M P L E
When the mirror is perpendicular to the horizon,

In fig. 77, H L is the horizontal line; S is its center, and S I its distance, which are also the center and distance of the picture. E represents the side of a room perpendicular to the picture; C is another side parallel to the picture; also M is a third side oblique to the picture, and to the other two sides, C and E, and may be considered as composed of looking-glass.

To find the vanishing points of the reflection of lines situated in a plane perpendicular to the Mirror.

The floor F is perpendicular to the face of the Mirror M; wherefore its reflection will be perpendicular to the back of the said mirror (by § 136) and consequently will be a continuation of the said plane F; so that the vanishing line, H L, of the floor, will be the vanishing line of its reflection.

The mirror M cuts the floor F in i g, which produce to its vanishing point V (*Vide* § 88) and because the mirror (M) is perpendicular to the horizon, draw V I perpendicular to H L; then is V I the vanishing line of the said mirror, (*Vide* § 104, page

184) next (by prob. 10, page 173) find the vanishing point (L.) of all lines perpendicular to the mirror. Then it is evident (by § 133) that the representation (1) of the reflection of any point (A) will be in a line (AL) drawn from the representation (A) of that real point to L. Draw a line through I and V, which will be the distance of ig ; and if the point I is supposed to correspond with i (*Vide* page 200) then I G will correspond with $i.g$. Through I draw l a parallel to H L; then does l correspond with A i , and the angle $a I G$ is equal to the angle which the line represented by A i makes with the face of the mirror M (*Vide* page 199 and 200;) wherefore draw $I a$, making with the back of the mirror the angle $G I a = G I a$, then is $I a$ the reflection of $a I$ (by § 135) wherefore producing $I a$ to v , I obtain the vanishing point of the reflection of A i , or of any line parallel to it (by § 138 and 17.) In like manner, if through any point a , a line $a b$ is drawn perpendicular to $I a$, it may be esteemed as corresponding with A B (*Vide* page 199 and 200;) wherefore if through I, $I m$ is drawn parallel to $a b$, the angle $m I G$ is equal to the angle which A B makes with the face of the glass M. Draw $I n$, making with $I G$ the angle $G I n = m I G$; then if $I n$ is produced till it cuts the horizontal line H L, we obtain the vanishing point of the reflection of A B, and of all lines parallel to it.

Another Method.

Let I (fig. 77) be supposed to correspond with the any point *i* in the figure whose reflection is to be represented. First draw a figure similar, and corresponding to it (*Vide* page 199 and 200;) thus I a, corresponds with *i* A; a b with A B; and because I can see the reflection of objects in the mirror M, which are hid by the side of the room E (the door D being open) I suppose c b to correspond with the side of a room adjoining to the point B, and parallel to A i: in this side b c, I suppose there is a door into the street; also at the point c, I suppose there is another side of the room, c d, running parallel to a b. Having thus obtained a figure similar to the real figures, whose reflections are to be represented, I suppose I G (which was found before) to be the face of the mirror; and I draw the several lines I a, a b, b c, c d, &c. of the same length, and in the same position to each other, and to the back of the mirror I G, as their correspondents I a, a b, b c and c d, are to the face of the mirror. Then regarding this reflected figure as a real figure, similar and corresponding to that whose representation is wanted, we may proceed as taught in problem 11, page 216.

Having thus found the reflection of the floor of the two rooms, every thing else is easily determined; because the corners

ners of the room, and the edges of the door, are parallel to the mirror; and the several boundaries of the ceiling are parallel to those of the floor. But to be more particular, v is the vanishing point of $I a$; (which is the reflection of $I a$) draw $i v$, and cut it in i by a line passing through A and L (*Vide* § 133) then is $i i$ the reflection of $i A$. Through i draw $i n$ parallel to $A N$ (*Vide* § 139) and through O draw $O v$, cutting $i n$ in n ; then is $i O i n$ the reflection of the parallel side of the room $i O N A$. The vanishing point of $a b$ falls beyond the bounds of the plate. Wherefore through any point (K) in the line $A B$, $K l$ is drawn parallel to $i A$, and cutting the mirror M in l ; then $l v$ is the infinite representation of the reflection of $K l$ (by § 138) and a ruler laid by K and L gives k , for the reflection of K (*Vide* § 133) wherefore since i is the reflection of A ; and k , that of K , $i k$ is the reflection of $A K$. Let a ruler be laid by B and L , cutting $i k$ in 2 ; then is $i 2$ the reflection of $A B$. Draw $2 p$ parallel to $B P$ (*Vide* § 139) and lay a ruler by P and L , cutting $2 p$ in p . Lastly draw $n p$; then is $i 2 p n$ the reflection of the perpendicular side of the room $A B P N$. How the reflection of the door, the inner room, and the bird-cage (the sides of which are parallel and perpendicular to the mirror) are determined will easily appear from inspecting the figure.

E X-

the vanishing line of the mirror M. Having thus got the vanishing line of the mirror

EXAMPLE II.

When the mirror is oblique to the ground.

In fig. 78, the same objects are represented as in the last figure; but the mirror (M) instead of being perpendicular to the ground, is now inclined to it; whereby, instead of cutting the parallel side (C) of the room in a line perpendicular to the horizontal line (H L) as in the last example, it now cuts the said parallel side of the room in the line $i O$, which is inclined to the horizontal line. Let it also be noted, that S is the center, and I S is the distance of the horizontal line (H L) and also of the picture. Our first operation must be

To find the vanishing point of such lines as are perpendicular to the Mirror (M).
 Fig. 78 is the section of the floor (F whose vanishing line, H L, is given) with the mirror M produce this till it cuts H L in V, which is its vanishing point by § 88, also because $i O$ is the section of the mirror with the parallel side (C) of the room, the said $i O$, must be the representation of a line situated in the plane of the mirror, and parallel to the picture; consequently (by § 54) the vanishing line of the mirror M must be parallel to $i O$, and it must pass through V (by § 87); wherefore V R drawn through V parallel to $i O$, is the

the vanishing line of the mirror M. Having thus got the vanishing line, VR of the mirror, draw from the center of the picture (S) SZ perpendicular to VR (by E. 1. 12) and cutting it in its center Z (Kide § 97), then the point P , which is the vanishing point of all lines perpendicular to the mirror, may easily be found (by prob. 10, page 173.) Our next business must be

To find the vanishing line of a plane perpendicular to the mirror, and passing by any line whose representation is given.

$C A S E I$

If the given line (as BA fig. 78) is not parallel to the picture,

Produce BA to its vanishing point S , and draw PS ; then is PS the vanishing line of a plane perpendicular to the mirror; for the plane passing by the lines represented by AB and BP is perpendicular to the mirror (by E. 1. 18.) and PS is its vanishing line (by § 89, page 169.)

$C A S E II$

When the given line is parallel to the picture, as A (fig. 78.)

Through P draw PV parallel to A ; and it is the vanishing line of a plane perpendicular

lar to the mirror, and passing by $A i$. For draw $A P$, which is the representation of a line perpendicular to the mirror (by the last operation) wherefore the plane passing by the lines represented by $A i$ and $A P$, is perpendicular to the mirror M , (by E. 11. 18.) and the vanishing line of that plane must pass through P ; (by § 87) and must be parallel to $A i$ (by § 54.)

We may now proceed

To find the reflection (i i) of any line ($A i$) whose representation and section (i) with the mirror, together with the vanishing line ($H L$) of the plane (F) in which it is situated, are given.

Through A , the extremity of the given line, draw $A P$, and through P draw at pleasure any line $P R$, cutting $H L$ in Q , and the vanishing line, $V R$, of the mirror in R ; through A , draw $A Q$, cutting $i g$ produced in q ; through q draw $q R$, cutting $A P$ in r ; then is r the point in which the perpendicular $A P$ cuts the mirror: For $P R$ may be considered as the vanishing line of a plane perpendicular to the mirror, because it passes through P ; also because it cuts $H L$ in Q , $A q$ is the section of the said plane with the floor (F); and $q r$ is its section with the mirror; wherefore the point r is in the surface of the mirror M ; and it is also in the perpendicular $A P$: consequently the point r is the section of
the

the said perpendicular $A P$, with the mirror M . We have now nothing more to do than to make $r i$ equal in representation to $A r$; for then the point r will be the reflection of the point A (by § 133.) To do this through P and A , draw any two lines, $P r$ and $A u$, parallel to each other, and cut them in the points t and u , by a line passing through r : make $u x = A u$, and draw $t x$, cutting $A p$ in i : then is $r i$ equal in representation to $A r$ (*Vide* § 84, page 123) and the point r is the reflection of the point A (by § 133;) and because the point i is situated both in the mirror M and the line in question $i A$, i is the reflection of $i A$. We come now to the fourth Operation.

The section ($i g$) of any plane (F) with the mirror, and its vanishing line ($H L$) being given, as also the reflection ($i r$) of some line ($A i$) situated in the said plane (F) to find the vanishing line ($V v$) of the reflection of that plane.

The reflection ($i r$) of the line $A i$ is in a plane passing by $A i$, and perpendicular to the mirror (by § 134) of which plane $P v$ is the vanishing line (*Vide* page 267); consequently the point v , where $i r$ when produced meets the said vanishing line $P v$, is the vanishing point of ($i r$) the reflection of $A i$. Having now obtained the vanishing point v of the reflection ($i r$) of a line $A i$ situated in the floor (F) and having also given the vanishing point V of

V of the reflection of g of another line (fig. 8) situated in the same plane of the floor (F) the line VV passing through these vanishing points, will be the vanishing line of the reflection of the floor, (Vide § 108, or any other horizontal plane, by § 107.)

(The center f , and distance d) of this vanishing line, may be found (by § 100, page 176) also the vanishing point (p) of such lines as are perpendicular to such planes as have V for their vanishing line, may be found by the latter part of prob. 10, page 173, and then we are sufficiently furnished for finding the representation of the reflection of the sides, ceiling and floor of the room, and of any object whose situation, with respect to either of these planes, is known; for, by reason of the inclined position of the mirror M, the position of all the reflected objects will be thrown into an inclined situation; but the magnitudes and positions of these objects to each other, must be esteemed the same as the real objects themselves (Vide § 103.) V is to be esteemed as the horizontal line, v is the vanishing point of the reflection of A, and of all lines parallel to it, (Vide § 108) wherefore f is the distance of that re-

A)
 For since g is situated both in the glass and in the floor, its reflection will coincide with itself.

lection

reflection. Through J draw JV perpendicular to JY , and cutting MY in V , then is V the vanishing point of the reflection of AVB and of all other lines perpendicular to AM . Also V is the vanishing point of the reflections of such lines as are parallel to the floor (F) and to the mirtour (M) (as are two sides of the bird cage) wherefore JV drawn through J perpendicular to JY gives V for the vanishing point of the reflection of such lines as are parallel to the floor (F) and perpendicular to the former (as are the other two sides of the bird cage); also the reflection of all lines that are perpendicular to the floor (F) as the edges of the door at the corners of the room, the line by which the cage hangs, &c.) tend to V . The figure will supply any other explanation: only let it be remembered, that the reflection (2) of any real point (B) is in a line (BP) drawn through that point, and the vanishing point (P) of lines perpendicular to the mirtour (M). *Vide* § 33, page 257.

N. B. When we have found the distance JV of the reflection of one side (AV) of the room, if we suppose J to coincide with one extremity (a) of that side of the room, and any point (a) in JV with the other extremity (A *Vide* page 200) we may then construct a figure similar to the ground plan of the several objects whose reflections we want to represent, taking care that the several lines are drawn in the same

posi-

positions respecting J a, as the real lines which form the ground plan are in respect of A i. Then will this similar figure enable us to find the reflections of the several horizontal lines, and be very useful if there are many complex objects. (*Vide* page 264.)

As these two examples comprehend the most difficult situations that can be assigned to the mirror, it will be needless to explain the method of determining the reflections of the objects by the water in fig. 79; for in this case, the reflecting plane being perpendicular to the picture, all lines and planes which are perpendicular to it, will be parallel to the picture; and consequently, to find the reflection of any point of an object, we have only to draw from that point a line perpendicular to the horizontal line (which is the vanishing line of the water or reflecting plane) and continue it so much below the surface of the water as the point, whose reflection is wanted, is above the said surface.

CHAP-

positions respecting λ , as the real lines which form the ground plan are in respect of λ . Then will this similar figure enable us to find the reflections of the several horizontal lines, and be very useful if there are many complex objects. (See page 264.)

As these two examples comprehend the most difficult questions that can be assigned to the student, it will be needless to explain the method of determining the reflections of the objects by the water in fig. 203 for in this case, the reflecting plane being perpendicular to the picture, all lines and planes which are perpendicular to it, will be parallel to the picture; and consequently, to find the reflection of any point of an object, we have only to draw from that point a line perpendicular to the horizontal line (which is the vanishing line of the water or reflecting plane) and continue it to reach below the surface of the water at the point whose reflection is wanted, as shown in the last figure.

CHAP.

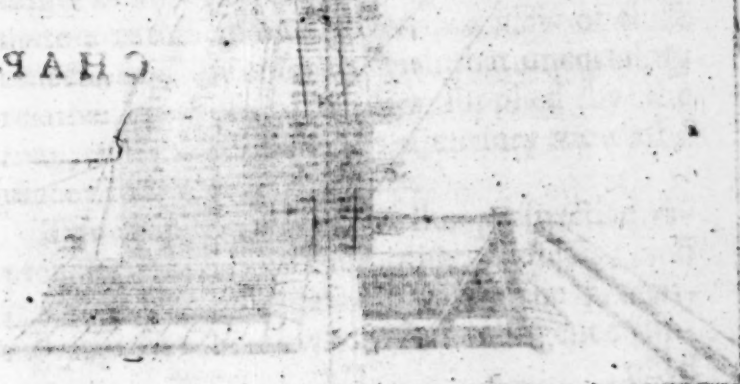
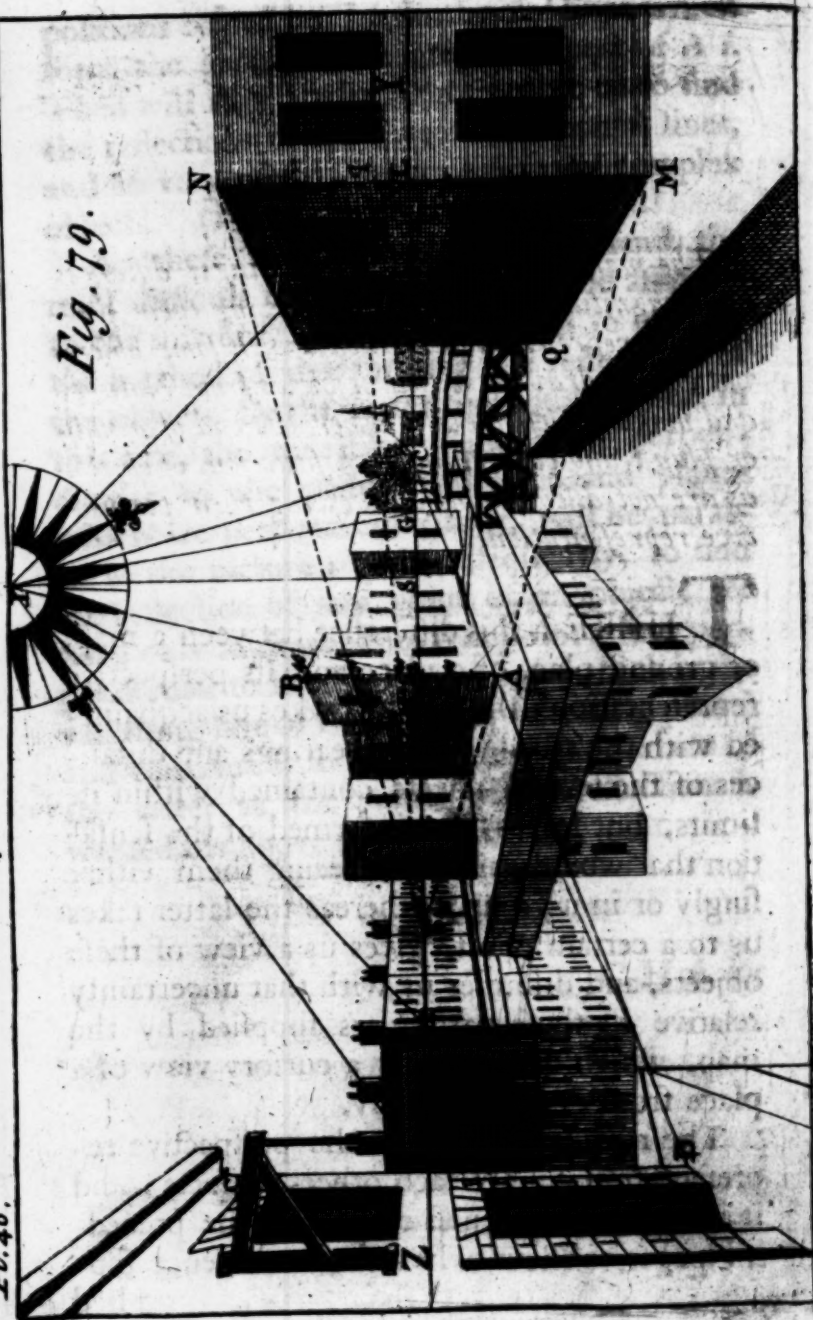


Fig. 79.

Pl. 46.





C H A P. XIII.

The manner of finding the bearings, distances and magnitudes of objects, by having their perspective representations given, with some observations for drawing extensive views from Nature, and making such views answer the end of a map or plan: also concerning such situations of objects as are not discoverable by viewing their perspective representations.

THERE is this difference between a map, or ground-plan, of a place and its perspective representation; the former makes us acquainted with the true situation, bearings and distances of the several objects contained within its limits, but leaves us uninformed of the sensation that would arise from seeing them either singly or in a group; whereas the latter takes us to a certain point, gives us a view of these objects, and dismisses us with that uncertainty relative to those particulars supplied by the map, which ever attends a cursory view of a place from one station only.

The map, or plan, and the perspective representation supply each others defects; and it has been the business of most of the preceding pages to shew, when the intelligence supplied

plied by the former is given or assumed, how the latter may be obtained ; to which end the heights of the several objects were also supposed to be determined either by actual mensuration or arbitrary assignment : And it will now be very easy to shew when the perspective representation, with the center and distance of the picture (by which such representation was obtained) are given, how to obtain the ground plan, as also the relative heights of the several objects represented.

P R O B L E M I.

CXL. *A perspective drawing with its center and distance being given, also the bearing (by the Compass) of any point in the picture from the eye, to construct a compass that shall shew the bearing of all the objects in the picture from the eye, and from each other.*

In this and the following Problems I shall confine myself entirely to the upright picture : fig. 79 represents a perspective drawing of a *Landscape*. S is the center and SI the distance of the picture. Also when the drawing was made, I suppose the artist took with a compass the bearing of the nearest corner (A) of the castle which was North East.

This being agreed, through A draw a line, A B, perpendicular to the horizontal line, H L, and cutting it in a ; then draw I a, and describe a circle about the center
(I)

(1) and construct a small compass * making the N E point to coincide with I a. Then will this compass inform us of the bearings of every object in the picture from each other, and also from the station where the eye was supposed when the view was drawn. Thus by drawing rumb lines from the center of the compass to the points where any perpendicular object cuts the horizontal line, we obtain the bearing of the said object from the eye; so we find the corner (C D) of the house (Z) bears from the eye E $\frac{1}{4}$ N. The corner of the street, E, bears E b N: the corner of the castle G bears N N E; and the corner of the house L, bears N b W, &c. If we would know how any of these objects bear from any point in the picture; as if I suppose myself placed at A, and I want to know how the several objects in the picture bear from that station; we must know the sections of the several objects whose bearings we want, as well as of the station from which those bearings are to be investigated, either with the ground or with some plane parallel to it. This being known, I lay a ruler by the station A, and the section of the object whose bearing from

* For the construction and names of the points of the compass, see any initiatory treatise on geography or navigation.

it I want; and I mark where it cuts the horizontal line; by which point, and the center of the compass, I lay a ruler; and it will shew the bearing required, having regard to the observation, page 200. Thus a ruler laid by the points D and A cuts H L in c; by which point, and the center of the compass (I) if a ruler is laid, I find the corner of the castle A B, and the corner of the house C D bear N b E $\frac{1}{2}$ E and S b W $\frac{1}{2}$ W from each other; and by the observation, page 200, it is evident that the corner of the house C D bears from the station A, S b W $\frac{1}{2}$ W.

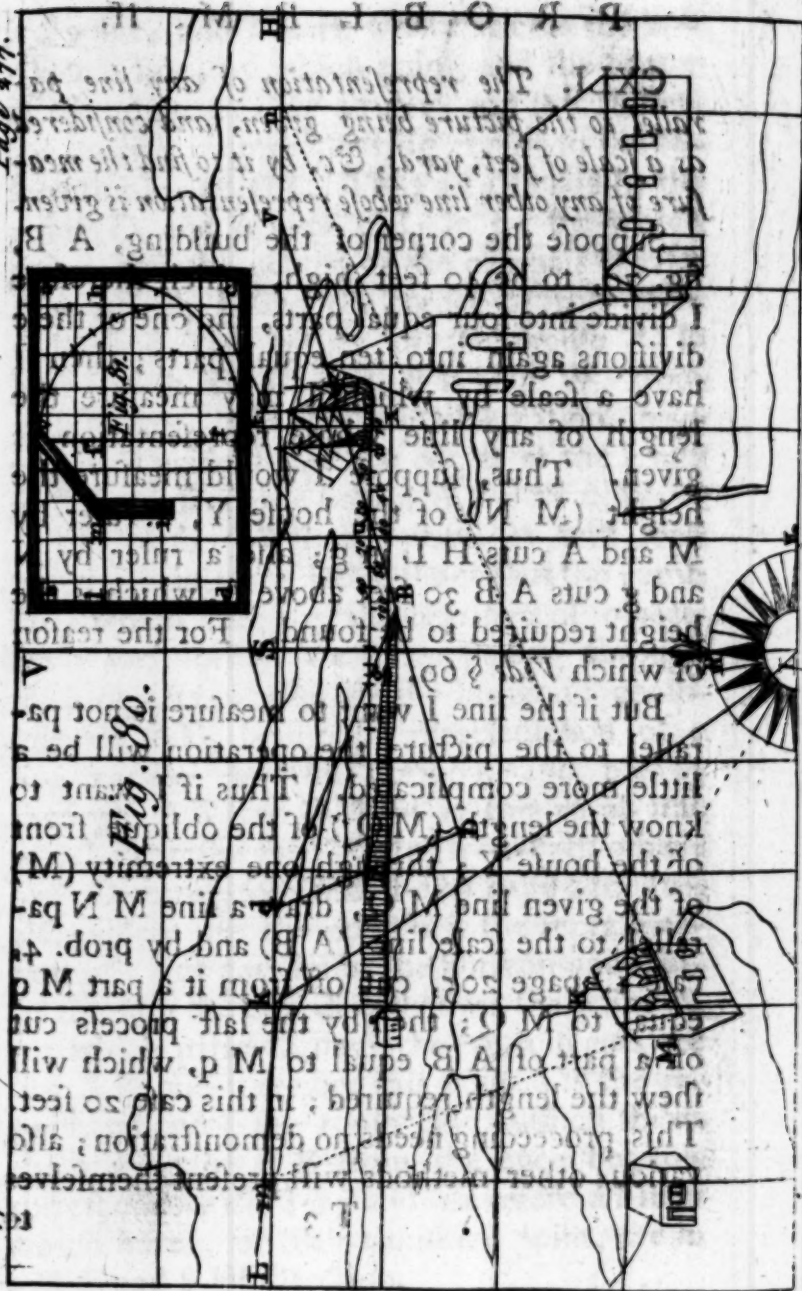
The reason of these operations will soon become evident, if we consider that the rumb lines which measure the bearings of objects from each other, are lines passing through those objects parallel to the horizon: consequently c D, being the representation of a line parallel to the horizon, and passing through A B and C D, it is the rumb line which expresses their bearing from each other: but c D corresponds with I c by page 200. It is also evident that the point a, is the representation of a line parallel to the horizon and passes through the eye, directly over the point A. (or which passes through the eye, and cuts the line A B) and therefore this point a is the representation of the rumb line, which shews the bearing of A B from the eye; but this corresponds with I a; and therefore all lines which have a, for their vanishing point, are in a N E and S W direction.

P R O B L E M

it I want; and I mark where it cuts the horizontal line; by which point, and the center of the compass, I lay a ruler; and it will shew the bearing required, having regard to the observation, page 200. Thus a ruler laid by the points D and A cuts H I in c; by which point, and the center of the compass (I) if a ruler is laid, I find the corner of the castle A B, and the corner of the house C D bear N b E & E and S b W: W. from each other; and by the observation, page 200, it is evident that the corner of the house C D bears from the station A, S b W: W.

The reason of these operations will soon become evident, if we consider that the turn lines which measure the bearings of objects from each other, are lines passing through those objects parallel to the horizon; consequently c D, being the representation of a line parallel to the horizon, and passing through A B and C D, it is the turn line which expresses their bearing from each other: but c D corresponds with c by page 200. It is also evident that the point a, is the representation of a line parallel to the horizon and passes through the eye, directly over the point A. For which passes through the eye, and cuts the line A B, and therefore this point a is the representation of the turn line, which shews the bearing of A B from the eye; but this corresponds with c, and therefore all lines which have, for their vanishing point, are in a N E and a W direction.

P R O B L E M



P R O B L E M II.

CXLI. *The representation of any line parallel to the picture being given, and considered as a scale of feet, yards, &c. by it to find the measure of any other line whose representation is given.*

Suppose the corner of the building, A B, fig. 79, to be 40 feet high, which therefore I divide into four equal parts, and one of these divisions again into ten equal parts; then I have a scale by which I may measure the length of any line whose representation is given. Thus, suppose I would measure the height (M N) of the house Y, a ruler by M and A cuts H L in g; also a ruler by N and g cuts A B 30 feet above A, which is the height required to be found. For the reason of which *Vide* § 69.

But if the line I want to measure is not parallel to the picture the operation will be a little more complicated. Thus if I want to know the length (M Q) of the oblique front of the house Y; through one extremity (M) of the given line M Q, draw a line M N parallel to the scale line (A B) and by prob. 4, case 1, page 205, cut off from it a part M q equal to M Q; then by the last process cut off a part of A B equal to M q, which will shew the length required; in this case 20 feet. This proceeding needs no demonstration; also various other methods will present themselves

T 3 to

to the practitioner when occasion renders them necessary.

CXLII. The two foregoing Problems will teach us to draw extensive views of bays, rivers, head-lands and capes, in a manner perfectly agreeable to the rules of perspective, and which shall afford such information of their bearings and distance from each other, and also of their appearance, as shall be vastly superior to any map or common drawing considered separately; and is worthy the attention of gentlemen in the sea service, who will hereby be enabled to make their accounts of the places they have visited much more useful, pleasing, and valuable, than is usually done.

To construct a simple Apparatus for taking extensive views with certainty and propriety.

For this purpose in fig 80 is a view of a river or harbour, in which are the several points and head-lands there exhibited; which view I suppose the artist draws in the following manner. Provide a square frame (a b c d fig. 81) which cross with two large threads h l and v r) parallel to the sides of the frame, one of which (v r) passes through its center, and the other (h l) cuts v r in f, which is about one third of v r from v; in the middle (v) of the side a b, let a hole be cut to fix in a joint (v m) perpendicular to the frame (a b c d) which

which joint may be of any length shorter than (v a) half the side of the frame. Also at m, another joint, m i, must be fixed to v m with a hinge, which will let it open so as to stand at right angles to v m. In this joint (m i) a hole must be made at i; so that when the whole is properly fixed, the hole i may be exactly opposite to the point f; for which reason m i must be equal to v f. Let v and h be considered as the points of contact of arcs having a b and a c for their tangents and lines equal to f i for their radii, and divide these sides a b and a c each way from v and h, into parts equal to the tangents of points, half points, and (if greater accuracy is sought after) quarter points of the compass. Through these divisions let threads be passed parallel to a b and a c; and, by this means, the frame will be divided into unequal squares, and is finished and fit for use; but it will be proper for greater distinction to make the threads which pass through every fourth division from v and h, (or which express whole points) large and black; those again which bisect these, and express half points smaller and red, and those which again bisect these last, and shew quarter points, still smaller and white.

When we want to draw any view with this instrument; on the paper allotted for that purpose draw two lines, H L and V R, fig. 80, crossing each at right angles in S, and divide these lines into tangents similar to the divi-

sions on the frame, assuming any radius (SI) for that purpose, which, by this means, becomes the distance of the picture. Then through these divisions draw lines dividing the paper into unequal squares similar to the frame, fig. 81. Let the artist now take his apparatus to some elevated station; * and placing his frame before him, he will, by looking through the sight i, see the objects he wants to draw depicted on the squares of his frame; which he may easily and accurately copy on his paper, by drawing in each square what is contained in the corresponding squares of the frame. Lastly, let him take a compass, and set the exact bearing of any object he has drawn; as suppose he observes the point K bears N W b W, then making SI equal to the distance of the picture, and drawing IK, we obtain the N W b W point of the compass, by which the other points may be drawn; from whence the bearings of the several objects from the eye, and from each other, may be determined by Prob. I; and if we can give a tolerable guess at the height which any object in the picture is above the surface

* If he is on board a ship riding at anchor in smooth water, he may repair to the topmast-head, and, standing on the cross trees, let him place his frame upright on the

of the water, such object may serve as a scale to measure the heights and distances of the other objects: thus if x is that point in the mainmast of the ship which is even with the surface of the water, and the truck (t) at the topgallant mast head is 60 yards above it, the line tx may be esteemed as a scale to measure all the distances. Thus through x draw ux parallel to HL , and equal to tx ; divide it into six equal parts, and continue the scale as the figure shews. Then to measure the distance of any two points, as B and D , cut off (by the last Problem,) a part of ux equal to BD : or, thus, produce BD till it cuts HL in y ; make vd equal to yl ; then laying a ruler by d and B , and D , it cuts ux in the points g and h , which by the scale are 100 yards asunder (*Vide* § 78.)

CXLIII. Since we can thus obtain the bearing of every object from any two stations (as from the place of the eye, and also from any point as K) and since we can also find the distance of any two of these objects from each other, any person who is the least conversant in the art of surveying, will not be at a loss how to plot these objects in a map or plan; from whence the true bearing and distances of them all from each other may be known.

Reason why the squares of the frame are not made equal.

CXLIV. Some may wonder that in the reticulated frame I make use of unequal divisions, formed

formed by tangents, instead of equal divisions which would answer the same end; but advantages arise from this, which the common method of making equal squares will not supply. To exemplify some of these, suppose I have the vanishing point *m*, fig 80, of one side of the roof of the house (*M*) I may easily find the vanishing point (*n*) of the other side, by taking *n*, eight points (or large divisions on the horizontal line) from *m*; because eight points are equal to a right angle; but if the fronts of the house were not perpendicular to each other, so many points, quarter points, &c. should have been reckoned from *m*, as might equal the inclination of the said fronts to each other. Again, because I know that most churches stand east and west, I make one side of the roof of the church tend to the north point in the horizontal line (which, in this example, corresponds with the center of the picture; and, of consequence, the other side of the roof which stands east and west, must be made parallel to the horizontal line.

If it should be objected that these uses of a perspective drawing cannot be understood by persons unacquainted with that art; I must observe that the same objection lies against all geographical maps and charts, which are useless if we know not the principles on which they are constructed. If a map is drawn by the principles of stereographic projection, the bearings and distances of the places

places laid down, cannot be obtained by the same means as we measure a map constructed by the orthographic projection. And each of these are different from the methods we are to pursue if the map is constructed by Mercator's projection, which differs again from the plane chart; for these reasons the scientific writers on navigation have thought it necessary to instruct the mariner in the principles of all these kinds of projection; and it is hoped that the art of perspective will, in time, be considered as a branch of science, with which a naval officer should not be wholly unacquainted.

It must be acknowledged, that, as objects recede to a great distance, their situations grows less and less perceptible; and the rules we have laid down will only furnish, with certainty, the bearings of the several objects from the eye; but this will suffice if we make two drawings from stations, whose bearings and distance from each other are mentioned; and if what is obscure and too far off at one station, is more at hand and distinguishable in the other; then will these two drawings supply all the data necessary to plot the several objects in a map.

I would not insinuate that the accuracy afforded by these methods is to compare with plans made by strict and careful surveys: and, at best, they are deficient in the most useful information which the others supply; namely,
the

the soundings or depths of water, which alone are to regulate the course of the vessel : but when the young artist, who is master of perspective, is furnished with a good plan, by the help of the foregoing observations, he may construct such views as will be most useful and ornamental companions for it, and entitle him to the favour and patronage of his superiors.

CXLV. From these methods of measuring the bearings and distances of objects represented in a perspective view, we come at the true knowledge of their situation, which we find, in some instance, to be very different from what we should apprehend from a cursory view of the drawing only. Thus I find that the oblique fronts of the houses Y and Z, on the right and left side of fig. 79, recede from each other ; so that the farther ends are more apart than the nearest ends ; which I should not have imagined from barely inspecting the figure ; on the contrary, an ignorant person would imagine they approach each other ; and even a connoisseur might, from viewing the outline of these figures alone, conceive the two fronts were parallel. This is a case in which I must own the *perspective representation* of an object gives an idea different from what arises by viewing the real object itself ; for I believe, if the real house Z was viewed from the situation, which is assigned to the eye in this scheme, the front would seem to recede from
the

the picture in an obtuse angle, as it really does; whereas its *perspective representation* leads us to conclude, that this front is either perpendicular or inclined towards the picture in an acute angle: but it must be remembered, that *Linear Perspective* only supplies the out-line of objects; and if this out-line was filled up with proper shades and colours, it would, I conceive, impart as perfect an idea of the true situation of the object represented, as arises from viewing the object itself from that situation which was assigned to the eye, when the said representation was determined by the rules of perspective.

CXLVI. The true situation of an upright plane reclining from the picture, is rendered less apparent because the representations of other objects do not contribute to point it out. But the case is very different in planes which recline from the picture below the horizontal plane, or which constitute what is called a *down-hill plane*; because objects posited on such planes acquire a form very different to what they would have if they stood upon level ground. In order to render this assertion evident, in fig. 72, are represented two similar rows of houses, one standing on level ground, and the other on a hill descending from the picture; the very different aspect of which sufficiently characterizes the inclined situation; and therefore I am obliged to dissent from a too popular opinion, "That it is impossible

possible to draw a down-hill view; because, say its advocates, such a view will impart to the mind a conception, that the objects are situated on an ascending plane; for if the vanishing line (V l) of the hill is higher than the bottoms of the houses, the representations of those bottoms (which tend to some point in the said vanishing line) must rise, as they do in this example; or if the said bottoms of the houses are above this vanishing line, then they will be hid, and a person who views the drawings will conceive them placed in a ditch. As this argument turns upon the *conceptions* which will arise from viewing such drawings, I confess myself unable to decide, unless I had heard the unbiassed opinion of great numbers of persons, on viewing some well executed drawings of this sort. But of this I am certain, from mathematical evidence, that the picture formed on the retina, (*Vide Art. 30 page 74*) by viewing the present scheme, has the same outline as the picture, which would be formed on the retina, by viewing a row of real houses situated on a descending plane; that to me, this representation does not appear to be the representation of a row of houses standing on an ascending plane, but the contrary; and that several persons to whom I have shewn a drawing of this kind rudely executed, have directly pronounced it to be a street going down hill. And I have no doubt but if any good artist was to make a painting or drawing

drawing of this kind, although the representation of the farther end of the descending plane would, in reality, be higher than that of the nearest end, the most ignorant person (who in this case would be the most proper judge) would pronounce it to descend, by the testimony arising from the dispositions of the objects which insist upon it.

P R O B L E M III.

CXLVII. *Having given the representation (A B, fig. 47) of a line divided in a certain point (d); also its vanishing point (V,) to find the proportion of its parts (A d and d B.)*

Through the points V and d draw any two lines (V L and a d b) parallel to each other; assume any point (D) in the line V L, and through it, and the extremities of the given line (A B) draw lines cutting a b in the points a and b. Then will a d and d b have the same proportion to each other as the originals represented by A d and d B.

D E M O N S T R A T I O N.

Because V L passes through the vanishing point (V) of A B, it may be considered as the vanishing line of a plane passing by A B; and because a b is parallel to it, it is the representation of a line parallel to the picture (*Vide* § 54) whose parts (a d and d b) have the same proportion to each other as the originals
repre-

represented by them (by § 59). But the triangles $a d A$ and $B d b$, are the representations of similar triangles; because $A a$ and $B b$ represent parallel lines (*Vide* § 17) wherefore (by E.6.4) the original of $A d$: original of $d B$:: (original of $a d$: origin of $d b$::) $a d: d b$ (by E 5. 11) $Q E D$

P R O B L E M IV.

CXLVIII. *To draw a triangle similar to one whose representation is given; the vanishing line and its center and distance of the plane in which it is situated, being also given.*

Let $a b i$ (fig. 82) be the representation of a triangle situated in a *working plane*, whose vanishing line is $V L$, and I is the place of the eye (*Vide* page 24 and 199, &c.) produce the sides of the triangle to their vanishing points V, L and d , and draw the distance lines $I L, I V$ and $I d$: then supposing the point I to correspond with i (*Vide* page 200 and 202) if a line ($A B$) is drawn parallel to $I L$, cutting the other two lines any where, as in the points B and A ; a triangle, $I B A$, will thereby be formed similar to the triangle represented by $i a b$. For the reason of this operation, *Vide* Prob. 9, page 213, which is only the converse to this.

By the foregoing problems we may find the magnitudes and positions of all the principal objects represented in a picture; but we must
either

either know its center and distance, or have some other data which is primarily derived therefrom, before we can attempt their solution: the center and distance of any perspective drawing should, therefore, always be pointed out, if we would extend its utility farther than merely to amuse the eye. Yet this is very seldom done; and indeed many very fine and admirable productions would discover much impropriety if measured with the rigour of these Problems. The following Problems will, in many cases, help us to find the center and distance of the picture, when the angles and the proportions of the sides of some of the objects represented, are known.

P R O B L E M V.

CXLIX. *Having given the representation (A B, fig. 47) of a line divided into two parts, (A d and d B) and the proportion of those parts (A d and d B) to each other; to find the vanishing point of the said line.*

Through d draw, at pleasure, the line a d b, and make a d to d b as the part represented by A d is to that represented by d B. Draw a A and b B cutting each other in D, and through D draw V L parallel to a b; lastly, produce A B till it cuts V L in V, which will be its vanishing point.

This is only the converse of Problem III, page 287.

U C O R O L

C O R O L L A R Y,

CL. Hence if the representation of two divided lines are given, the vanishing line of the plane in which they are situated, may be found. *Vide* § 89.

P R O B L E M VI.

CLI. *The representation* (i b a, fig. 82) *of a triangle being given, with the vanishing line, V L, of the plane in which it is situated; as also the angles of the said triangle: to find the center and distance I S of the given vanishing line.*

Suppose i b a, to be the representation of a right-angled triangle, whose oblique angles represented by b a i and b i a, are each equal to 45° .

Produce the sides of the given triangle to their vanishing points V, d and L; then by E. 3. 33. on V d describe the segment of a circle, which shall contain an angle equal to that represented by b i a $= 45^\circ$. Also on d L describe the segment of a circle, which shall contain an angle equal to that represented by b a i, (which in this case is also 45°); then from the point (I) where these two arcs cut each other, draw I S perpendicular to V L; then is S the center, and I S the distance of the given vanishing line V L.

D E M O N-

DEMONSTRATION.

In § 32 it was demonstrated, that when the place of the eye (I) was laid down on the picture, the distance lines (I V and I d) of any visuals (i V and i d) make the same angle with each other as the original line represented by those visuals; wherefore it is evident (from the construction and E. 3. 21) that the place of the eye (I) must be in the circumference of both the circular segments; consequently it must be in the point (I) where those segments cut each other. Q. E. D.

COROLLARY.

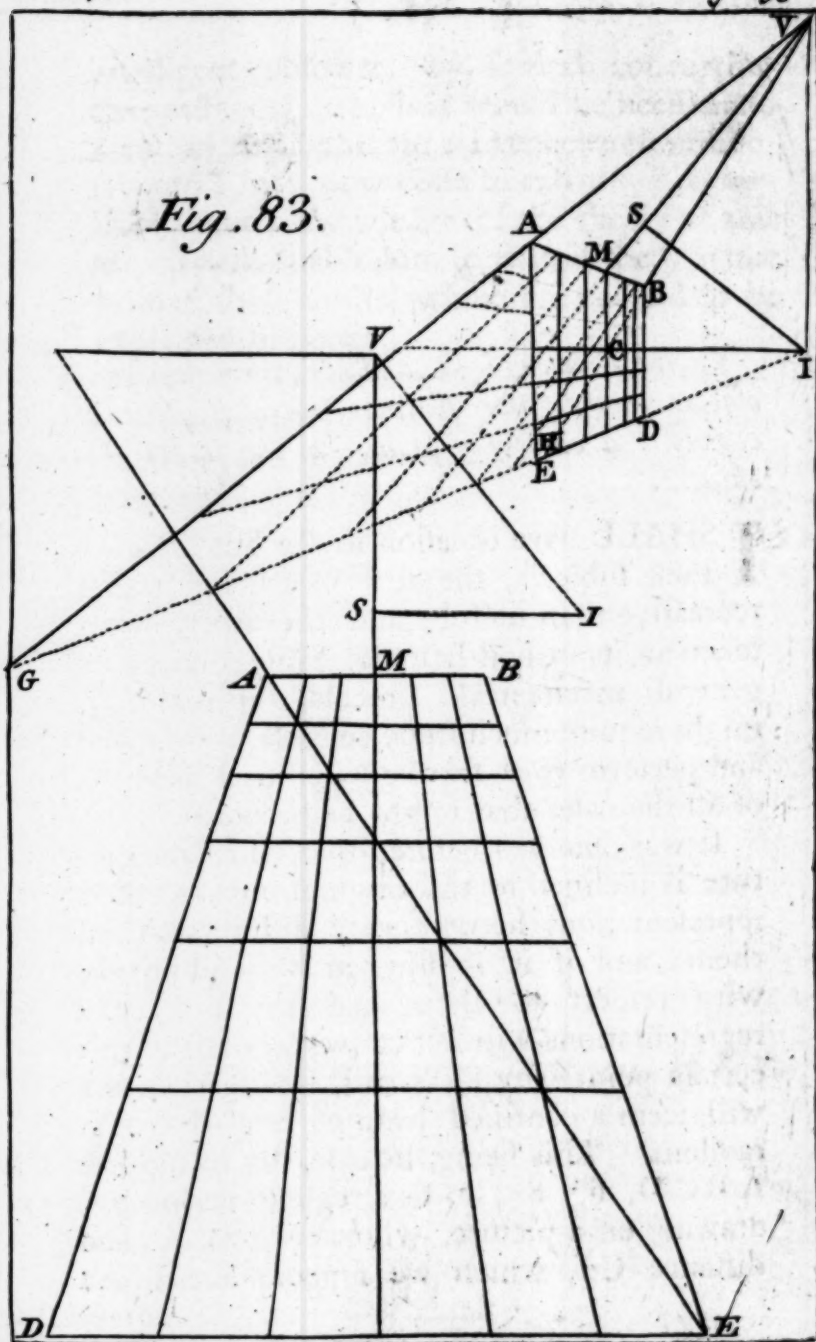
CLII. If the representation of two sides of a square had been given, it is easy to perceive how it might have been resolved into two such right-angled triangles as is represented in this example; or if any irregular trapezium of a known similarity is given, it may be resolved into triangles, whose angles will be known; from whence the center and distance of the vanishing line may in like manner be determined.

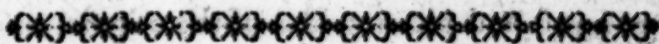
Any person versed in the theory and practice of perspective, will easily find the horizontal line of most drawings or paintings that fall under his observation; and, if the piece is executed with skill and propriety, its center and distance will be pointed out, to an

intelligent observer, by several concurring circumstances, which it would be needless to mention, as by this time I conceive the reader (whom I shall now cease to call the *Learner* has so much knowledge of the theory of this art, as will enable him to extend the practice beyond those limits, which are needful to be explained in books. -

CHAP.

Fig. 83.





C H A P. XIV.

Of the Anamorphosis; and the representation of objects on irregular surfaces composed either of curves or a variety of planes, as the scenes of a theatre.

I SHALL have occasion to say but little on these subjects, the first part being rather recreative than useful; and the other, when taken in its utmost latitude, easily solved by a general mechanical process, although it might require much study and labour to invent and perform rules for the scientific solutions of all the cases that might be proposed.

It was observed before, that when the picture is inclined to the original objects, their representations become very dissimilar from them; and if it is situated very obliquely with respect to them and the eye, the representations will not convey (unless in one certain point) any ideas of those objects; but will seem a confused heap of lines drawn at random. This being the case, let us suppose $ABCD$, fig. 83, to be a regular perspective drawing on a picture, whose center is C , and distance CI , which we suppose determined agree-

agreeable to Chap. VI: it is plain that if this picture is viewed by an eye at I, it will convey a just and pleasing idea of the original objects which are represented on it. Let this picture, which we call the *Prototype*, be divided into any number of small squares; and by the top A B of the prototype, let a plane G H V be passed at pleasure, so as to have any obliquity to the said prototype. Then if the rays which come from the sides of the squares drawn on the prototype, to the eye at I, are continued till they cut the oblique plane G H V, they will divide that part of it, A B G H, which is beyond the prototype, into the like number of irregular trapeziums, (as are expressed by the dotted lines) which will be more or less unequal and irregular, according as the plane G H V is more or less oblique to the Prototype. Let whatever is contained in each square of the prototype, be drawn in its corresponding trapezium on the plane G H V; so will the whole trapezium A B G H contain a distorted representation of the regular picture A B C D, but which to the eye, at I, cannot be distinguished from that regular picture or prototype. This representation may be obtained two ways: the first by the rules of perspective; for if we consider the inclined plane G H V as the picture, and I as the place of the eye; then a perpendicular I S drawn from the eye to the picture, will give S for its center, and S I for its distance; and if the pro-
 totype

prototype *A B C D* is considered as an original plane, whose entering line is *A B*, *V I* being drawn through the eye parallel to the perpendicular side of the squares of the prototype, will give *V* for their vanishing point, and *V I* for their distance. We are now furnished with all the requisites for representing the squares of the prototype in perspective, as is exemplified by the lines distinguished by the *Italic Capitals*, which exactly correspond to those marked by the *Roman Capitals* already described. To make this distorted representation appear regular, we must raise at *S* a line, or small stick, having at top a plate with a small hole in it, perpendicular to the plane of the paper, and equal to *S I* or *S I*; then provide a square frame of the same size with the prototype *A B C D*; place one side to coincide with *A B*, and incline it to the plane on which the deformed representation is drawn, in an angle equal to *S V I*; then if viewed through the hole already described, the deformed representation, or anamorphosis, will exactly fill the square frame, and appear as regular as the prototype *A B C D*.

It must be observed that the distance of the eye, *I*, from the line *A B*, that is, the line *I M* or *I M*, ought never to be less than six inches; for if it is, as in our plate, the eye, when placed to view the anamorphosis, will not have distinct vision of the nearest parts of it, the rays at so small a distance diverging too much to be united

united on the retina by the humours of the eye. *Vide* page 75.

The second method to find the irregular trapeziums which answer to the squares of the prototype, is more general and easy, although wholly mechanical; for, instead of the prototype, let us suppose a frame of the same dimensions exactly divided into the same number of small squares, by means of threads crossing each other, to be put in the place of the prototype $A B C D$; and at I , where the eye was supposed to be situated, let the flame of a lamp be placed (which is better than a candle, because it will not alter its situation by burning lower) then it is evident the shadows of the threads on the frame $A B C D$, will produce on the inclined plane $G H V$, those irregular trapeziums which are the representations of the squares on the prototype, which may be traced over, and filled up as before directed.

- If instead of a plane, as $G H V$, we have any other kind of surface, either concave or convex, or mixed and comprized of several concave, convex and plain surfaces joined together in a manner ever so irregular, it is easy to conceive, that the shadows of the threads will divide this irregular surface into spaces, which answer to the squares on the prototype.

N. B. Care must be taken if the prototype is a perspective drawing of various objects, that the flame of the lamp be placed in the true point

of sight which the eye ought to be in to view that drawing, agreeable to what was observed before.

This thought, pursued a little farther, may hint an easy method of drawing the scenery of theatres; for if the prototype is made to consist of such a *perspective representation*, as is to result from the whole scenery put together, (in the drawing of which prototype the curtain may be considered as the picture, and the eye may be supposed situated just above the middle of the front boxes, or the bottom of the lower gallery) and if a reticulated frame is placed in lieu of the curtain, behind which planes are to be situated in such a manner as may be most convenient for the action of the Drama; and if a lamp is placed in the situation chosen for the eye when the prototype was drawn, the shadows of the squares of the reticulated curtain will mark on the several planes placed behind it (which are the scenes) the spaces which are to be filled up from the prototype first drawn. This method, perhaps, will be more practicable if a model of the stage, and the situation chosen for the eye, is made on a small scale; because, in the real theatre, the shadows of the threads will be indistinct, by reason of the great distance which the front boxes (where the lamp is to be placed) are from the back part of the stage where the most remote flat scene, which bounds the whole view, must be. This method,

X

assisted

assisted with the theory of vanishing points and lines, explained in Chap. VIII. will enable the reader to amuse himself with disposing a number of perspective representations together, in such a manner as to afford the most beautiful and natural appearances imaginable. There is a method still better than this. If when the prototype is drawn, we also mark out thereon the perspective representations of that part of the stage on which the action is to be performed, and also draw the representations of the planes, which are to receive the scenery according to the situation in which they must be placed on the stage; then it is evident the whole prototype will fall upon one or other of these represented planes; and if on the real planes, or scenes, the images are copied which fall on the representations of those planes, according to the measures and positions which may be obtained by the last chapter; the whole will, when set up in their proper places, paint on the retina of the eye the same picture as is formed by viewing the prototype from the proper point of sight. I only offer these hints for the intelligent, whose business or pleasure may incline them to representations of this kind; and who will be able to understand me without a long train of words and schemes, which would only enhance the bulk and price of this work with what is not generally useful.

F I N I S.



